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NOTE ON SETS OF POSITIVE MEASURE*

BY HENRY BLUMBERG

A recurring question concerning (L-measurable) sets of positive measure is what properties they have in common with the linear interval. The following theorem is concerned with such a property, stated for sets of n-dimensional positive measure lying in euclidean n space.

THEOREM. *Let $A_1, A_2, \cdots, A_p$ be p sets of positive measure lying in euclidean n space. Then there exist p n-dimensional spheres $S_1, S_2, \cdots, S_p$ such that for every set of p points s_ν, $(\nu=1,2, \cdots, p)$, belonging respectively to these spheres, there exists a set of p points a_ν, $(\nu=1, 2, \cdots, p)$, lying respectively in A_1, $A_2, \cdots, A_p$, such that the sets $\{a_\nu\}$ and $\{s_\nu\}$ are congruent. Moreover, there exists a set of p congruent spheres S_ν satisfying the condition just stated and a positive number δ such that for every selected $\{s_\nu\}$, with s_ν belonging to S_ν, the associated $\{a_\nu\}$ may be so chosen that a_1 ranges over a set of measure $>\delta$.*

PROOF. Since A_ν is of positive measure, there is a sphere S_ν' in which the relative measure of A_ν is greater than $1-\epsilon$, where ϵ is a given positive number less than 1; that is, $m(A_\nu, S_\nu')/m(S_\nu') > 1-\epsilon$, $m(A)$ standing for the measure of A. We may suppose, and we do so for simplicity of statement, that all the S_ν', $(\nu=1, \cdots, p)$, are equal, and we denote their common measure by μ, and their respective centers by c_ν. Let ρ be a positive number such that if a sphere of measure μ is translated a distance $<\rho$, the part belonging to the sphere in both positions is of measure $>(1-\epsilon)\mu$. Denote by v_ν, $(\nu=1, \cdots, p-1)$, the vector represented by the segment $c_\nu c_{\nu+1}$; and let w_ν, $(\nu=1, \cdots, p)$, be a given set of n-dimensional vectors, each of length $<\rho$. If a set A (or point a) is given a displacement represented by the vector $\pm v$, we denote the set (or point) in its new position by $A \pm (v)$ (or $a \pm (v)$). Writing $A_\nu S_\nu' = T^{(\nu)}$ and $T' = T_1'$, we set

$$T_1' + (v_1 - w_1 + w_2) = T_2'; \; T_2' T'' = T_1'';$$

* Presented to the Society, April 14, 1923.

$$T_1'' + (v_2 - w_2 + w_3) = T_2' \; ; \; T_2'' T''' = T_1''' ;$$

. . . . . .

$$T_1^{(p-1)} + (v_{p-1} - w_{p-1} + w_p) = T_2^{(p-1)} ; \; T_2^{(p-1)} T^{(p)} = T_1^{(p)} .$$

For the measures of the $T_\lambda^{(\nu)}$'s, we have the following inequalities: $m(T_1') > (1-\epsilon)\mu$; $m(T_1'') > (1-4\epsilon)\mu$, since $m(T'') > (1-\epsilon)\mu$ and the lengths of w_1 and w_2 are less than ρ; $m(T_1''') > (1-7\epsilon)\mu$, and so on. We may thus conclude that $m(T_1^{(p)}) > [1-(3p-2)\epsilon]\mu$, which is positive if ϵ is taken small enough. We now define the spheres $S_1, \cdots, S_p$ of our theorem as of radii all less than ρ, and such that their respective centers γ_ν satisfy the relations: vector $\gamma_\nu\gamma_{\nu+1} = v_\nu$, $(\nu = 1, \cdots, p-1)$. If now s_ν, $(\nu = 1, \cdots, p)$, is a point chosen from S_ν, we let vector $\gamma_\nu s_\nu = w_\nu$. Let a_p be a point of $T_1^{(p)}$, which, as we have seen, is not empty if ϵ is sufficiently small. We then define $a_{p-1}, a_{p-2}, \cdots, a_1$ by the relation

$$a_\nu = a_{\nu-1} + (v_{\nu-1} - w_{\nu-1} + w_\nu), \qquad (\nu = 2, 3, \cdots, p).$$

Since a_p belongs to $T_1^{(p)}$, it belongs to A_p and also to $T_2^{(p-1)}$; hence a_{p-1} belongs to $T_1^{(p-1)}$ and therefore to A_{p-1} and $T_2^{(p-2)}$; hence a_{p-2} belongs to $T_1^{(p-2)}$, and so on. We conclude that a_ν, $(\nu = 1, \cdots, p)$, belongs to A_ν. Since s_ν satisfies the equation $s_\nu = s_{\nu-1} + (v_{\nu-1} - w_{\nu-1} + w_\nu)$, $(\nu = 2, \cdots, p)$, we see that the sets $\{a_\nu\}$ and $\{s_\nu\}$ are congruent. Furthermore, since a_p is an arbitrary point of $T_1^{(p)}$, whose measure is arbitrarily near μ, we can satisfy the last condition of our theorem by taking, for example, $\delta = \mu/2$, if ϵ is small enough.

If, in particular, the p given sets A_ν are identical, we may take the spheres S_ν' as identical, thus reducing the vectors v_ν to zero. The spheres S_ν may therefore be taken as identical, and we have the following corollary.

COROLLARY. *If A is an n-dimensional set of positive measure, there exists an n-dimensional sphere S, such that for every finite subset of S there is a congruent subset of A.*

If A is a one-dimensional set, we obtain the theorem of Steinhaus,* that *the set of distances between pairs of points of a*

* *Sur les distances des points*, Fundamenta Mathematicae, vol. 1 (1920), p. 99. A simpler proof of this theorem, close in idea to our own, was published by Ruziewicz (after the present paper was read), Fundamenta Mathematicae, vol. 7 (1925), p. 141.

(linear) set of positive measure contains an interval with 0 as left end point.

We have proved that if S_1 is a set of positive measure and S_2 a finite set, there is a subset of S_1 similar* to S_2. To what extent can the condition of finiteness of S_2 be modified if the theorem is to remain valid? Since every set of positive measure contains a perfect subset of positive measure, it follows that if every set of positive measure contains a set similar to S_2, it contains a set similar to S_2+S_2', where S_2' is the derivative of S_2. It thus suffices to restrict S_2 to being closed. Or we may restrict S_2 to being denumerable, since every set contains a denumerable subset which is dense in it. Not all sets of positive measure can contain a set similar to S_2 if S_2 is not non-dense. Since we naturally restrict S_2 to being bounded, we are led to ask: What bounded, non-dense sets S_2 are such that every set of positive measure contains a set similar to S_2? That this property is not shared by every bounded, non-dense S_2, and therefore not by every bounded, non-dense, denumerable set, is shown by the following fact.

THEOREM. *If S_1 is a given bounded, non-dense, perfect set, there exists a perfect set S_2 of zero measure such that no subset of S_1 is similar to S_2.*

While this theorem is meant to refer to n-dimensional sets, we assume in the proof that S_1 and S_2 are linear sets, there being no essential difference in the argument for n-dimensional sets. We suppose, as we may, that the given set S_1 lies in the interval $(0, 1)=I$. Let $C(S_1)$ be the complement of S_1 in I; Δ a variable subinterval of I; $\mu_1(\Delta)$ the ratio of the maximum length of a connected portion of $C(S_1)$ in Δ to the length of Δ; and $\sigma_1(h)$, for h a given positive number, the greatest lower bound of $\mu_1(\Delta)$ for all subintervals Δ of I of length h. Then $\sigma_1(h)$ is a positive, continuous function of h. The perfect set S_2 will be defined as the complement in I of the set of intervals Δ_{ni}, which are defined as follows: Insert in I a set of equally spaced intervals Δ_{1i}, $(i=1, 2, \cdots, m_1)$, of equal length l_1, such that $m_1 l_1 = 1/2$, the equality of spacing being understood in the sense that the

* We are using "similar" in the ordinary euclidean sense of the existence of a biunique correspondence with invariant ratio of distances.

space between any two adjacent Δ_{1i} shall be equal to the spaces between 0, 1 and the first, last Δ_{1i}, respectively; moreover, m_1 is to be so large that $\sigma_2(\lambda_1)<\sigma_1(\lambda_1)$, where $\lambda_1=1$, and σ_2 has the same meaning for the set S_2, now being defined, as σ_1 for S_1. Similarly insert in each of the intervals Δ'_{1i}, $(i=1, 2, \cdots, m_1+1)$, of length l'_1, that are complementary to the Δ_{1i} the same number of equally-spaced intervals Δ_{2i}, $(i=1, 2, \cdots, m_2)$, of equal length l_2, where m_2 signifies the total number of the Δ_{2i} in all the Δ_{1i}; moreover, the Δ_{2i} are to be such that $m_2 l_2=1/4$, and m_2 so large that $\sigma_2(t\lambda_2)<\sigma_1(\lambda_2)$ for $1\leqq t\leqq 2$, where $2\lambda_2=l'_1$. In general, let $\{\Delta'_{n-1,i}\}$ be the set of intervals of length l'_{n-1}, complementary to the set of all $\Delta_{\nu i}$, $\nu\leqq n-1$. Insert in each $\Delta'_{n-1,i}$ the same number of equally spaced intervals Δ_{ni} of equal length l_n, such that $m_n l_n=1/2^n$, m_n being the total number of intervals Δ_{ni} and m_n so large that $\sigma_2(t\lambda_n)<\sigma_1(\lambda_n)$ for $1\leqq t\leqq n$, where $n\lambda_n=l'_{n-1}$. Suppose now that S_3 is any set whatsoever lying in I and similar to S_2; then S_3 cannot lie in S_1. For let k be the ratio of corresponding lengths in S_2 and S_3, and n an integer greater than k. If ϵ is a given positive number, we can find an interval Δ of length λ_n lying between two points of S_3, and such that $|\mu_3(\Delta)-\sigma_2(k\lambda_n)|<\epsilon$, μ_3 having the same meaning for S_3 as μ_1 for S_1. Hence, on account of the inequality $\sigma_2(k\lambda_n)<\sigma_1(\lambda_n)$, we conclude that $\mu_3(\Delta)<\sigma_1(\lambda_n)\leqq\mu_1(\Delta)$, if ϵ is small enough. That is to say, the maximum length of a connected portion of $C(S_3)\Delta$ is less than such maximum length for $C(S_1)\Delta$, and therefore S_3 cannot lie in S_1.

THE OHIO STATE UNIVERSITY

NOTE ON THE GYROSCOPE

BY W. F. OSGOOD

By a *gyroscope* we mean a rigid body having an axis of material symmetry and free to turn about a certain fixed point O of the axis. Let the gyroscope be acted on by any forces, and let it be required to determine the motion of the axis under the action of these forces. By far and away the most interesting and important case is that in which the component couple tending to produce rotation about the axis is nil, so that the component angular velocity about the axis is constant; denote it by ν. Under this restriction the resultant of the applied forces can be expressed as a single force, $\mathfrak{F}$, acting at a point P on the axis at unit distance from O, normally to OP, and by the force the peg at O exerts.

Let $\mathfrak{C}$ be the curve P describes on the unit sphere about O. The problem is solved by the intrinsic equations:*

$$A\frac{dv}{dt} = T, \qquad A\kappa v^2 + C\nu v = Q,$$

where C and A are the moments of inertia about the axis and about a normal to the axis through O; T and Q are the components of $\mathfrak{F}$ along the tangent and normal (in the tangent plane of the sphere) of $\mathfrak{C}$; and κ is the *bending* of the cone determined by O and $\mathfrak{C}$,—or the rate at which the plane through O tangent to $\mathfrak{C}$ is turning when the point of tangency describes $\mathfrak{C}$ with unit velocity.

The object of the present note is to point out a simple interpretation of these equations in terms of the motion of a material particle carrying a charge of electricity and moving in the electro-magnetic field of force generated by the north pole of a magnet situated at O. Let the mass, m, of the particle be A, or $m = A$; and let the particle be constrained to lie on the unit sphere about O. Let

$$Q_1 = A\kappa v^2, \qquad Q_2 = C\nu v.$$

* See the author's paper, *On the gyroscope*, Transactions of this Society, vol. 23 (1922), p. 240.

Then it is easily shown that the particle without any electrical charge will describe the path $\mathfrak{C}$ if acted on by the tangential force T and the normal force Q_1; for the equations of motion then become

$$m\frac{dv}{dt} = T, \qquad m\kappa v^2 = Q_1.$$

Now apply a charge, e, of (positive or negative) electricity to m. The force which the field exerts on the charge will be proportional to the velocity, v, of the charged particle in its path, and in a direction normal to the path and to OP. By suitably choosing e this force can be made to equal precisely the negative of Q_2, or $-Cv$. If, then, the force which acts on the mass m is the same force $\mathfrak{F}$ that acts on the gyroscope at P, with its components T along the tangent and

$$Q = Q_1 + Q_2$$

along the normal, the electro-magnetic force of the field on the electrical charge, namely, $-Cv$, will precisely eliminate the term Q_2 in Q and leave only Q_1 to act on m.*

The result may be formulated in the following theorem.

THEOREM. *When the gyroscope is acted on by a force $\mathfrak{F}$ at P normal to the axis, the point P moves exactly as a particle of mass $m=A$ acted on by the same force $\mathfrak{F}$ and constrained to lie on the unit sphere about O would move if m were carrying a suitable charge e of electricity and the motion of m thus charged took place in an electro-magnetic field generated by the north pole of a magnet situated at O.*

HARVARD UNIVERSITY

* I am indebted to my colleague, Professor Kemble, for the suggestion of a radial electro-magnetic field and a charge of electricity moving in that field, as a means of taking care of that part of the force at right angles to the path, which corresponds to the term Cv.

NOTE ON A SPECIAL CYCLIC SYSTEM*

BY MALCOLM FOSTER

1. *Introduction.* This note is concerned with a special cyclic system.† Let S be a surface referred to any orthogonal system, and T the trihedral whose x-axis is tangent to the curve $v=\text{const.}$ The equations

$$x=R(1+\cos\theta), \quad y=0, \quad z=R\sin\theta, \tag{1}$$

define a two-parameter family of circles C normal to S; and the necessary and sufficient conditions that C shall constitute a cyclic system are

$$\xi\frac{\partial R}{\partial v}+R\eta_1 r=0, \quad R(pr_1-p_1 r)-q_1\left(\xi+\frac{\partial R}{\partial u}\right)+q\frac{\partial R}{\partial v}=0. \tag{2}$$

It is readily seen that the first of equations (2) may be written‡

$$\xi\frac{\partial R}{\partial v}-R\frac{\partial \xi}{\partial z}=0;$$

hence

$$R=U\xi, \tag{3}$$

where U is a function of u alone. Using (3) we may write the second equation of (2) in the form

$$U\xi(pr_1-p_1 r)-q_1\left(\xi+\xi U'+U\frac{\partial \xi}{\partial u}\right)-qU\eta_1 r=0. \tag{4}$$

We shall therefore replace equations (2) by (3) and (4).

2. *The Inversion of C.* If we invert the circles C relative to the circles $x^2+z^2=K^2$, $y=0$, where K is any constant, we get the following system of lines L,

$$x=\frac{K^2}{2R}, \quad y=0, \tag{5}$$

* Presented to the Society, March 25, 1932.

† See Eisenhart, *Differential Geometry of Curves and Surfaces*, Ex. 11, p. 444.

‡ Eisenhart, p. 170.

which will constitute a rectilinear congruence. Let us determine the condition that this congruence be normal.

Any point on the lines (5) will have the coordinates $(K^2/(2R), 0, z)$. The necessary and sufficient condition that (5) shall define a normal congruence is that $\delta z = 0$ for all values of dv/du. Hence

$$\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv - \frac{K^2}{2R}(qdu + q_1 dv) \equiv 0,$$

or

$$\frac{\partial z}{\partial u} = \frac{K^2 q}{2R}, \quad \frac{\partial z}{\partial v} = \frac{K^2 q_1}{2R}.$$

The condition of integrability is that $\partial^2 z/\partial u \partial v = \partial^2 z/\partial v \partial u$; hence

$$R\frac{\partial q}{\partial v} - q\frac{\partial R}{\partial v} = R\frac{\partial q_1}{\partial u} - q_1\frac{\partial R}{\partial u},$$

which by means of (3) becomes*

$$U\xi(rp_1 - r_1 p) + qU\,\eta_1 r + q_1\left(\xi U' + U\frac{\partial \xi}{\partial u}\right) = 0.$$

On making use of (4), this further reduces to $q_1 = 0$; hence we have the following theorem.

THEOREM 1. *A necessary and sufficient condition that the lines L, obtained by the above inversion of the cyclic system C, shall form a normal congruence, is that S be referred to its lines of curvature.*

We readily find that for $q_1 = p = 0$, (4) becomes $r(\xi p_1 + \eta_1 q) = 0$. Two cases are to be considered.

(a) If $r = 0$, the curves $v = \text{const.}$ are geodesics; and since they are lines of curvature, they are plane curves. Consequently the planes of C constitute but a one-parameter family, and their envelope is the developable surface which is readily seen to be one of the focal sheets of S. We note also from (2) that for $r = 0$, R is a function of u alone.

(b) If $\xi p_1 + \eta_1 q = 0$, it is readily seen† that $D:E = D'':G$. Hence S is in this case either a sphere or a plane. If S is a sphere the planes of C pass through the center.‡

* Eisenhart, p. 168.

† Eisenhart, p. 174.

‡ Eisenhart, p. 441.

3. *The Focal Points of C.* The direction cosines of an arbitrary tangent of C are $(-\sin\theta, 0, \cos\theta)$. Since the displacements $\delta x, \delta y, \delta z$ of the focal points of C must be in the direction of the tangents we have $\cos\theta\ \delta x + \sin\theta\ \delta z = 0$, $\delta y = 0$. On using (1), these equations may be written

$$\begin{aligned} &\frac{x}{R}\frac{\partial R}{\partial u}du + \frac{x}{R}\frac{\partial R}{\partial v}dv + \frac{\xi(x-R)}{R}du - z(qdu + q_1dv) = 0, \\ &\eta_1 dv + x(rdu + r_1dv) - z(pdu + p_1dv) = 0. \end{aligned} \tag{6}$$

Obviously, we must have

$$\begin{vmatrix} \dfrac{x}{R}\dfrac{\partial R}{\partial u} + \dfrac{\xi(x-R)}{R} - qz & \dfrac{x}{R}\dfrac{\partial R}{\partial v} - q_1 z \\ rx - pz & \eta_1 + r_1 x - p_1 z \end{vmatrix} = 0. \tag{7}$$

Hence the focal points of C, which are at most four in number, are given by (1) and (7).

Let us now consider the case when the inversion of C leads to a normal rectilinear congruence. We have seen that we must have $p = q_1 = 0$, and $r(\xi p_1 + \eta_1 q) = 0$. The first case, (a), is of some interest; for $p = q_1 = r = 0$, equation (7) becomes*

$$\left(\frac{\partial R}{\partial u} + \xi\right)x - Rqz - R\xi = 0, \qquad r_1 x - p_1 z + \eta_1 = 0. \tag{8}$$

We note that the two centers of principal curvature, $(0, 0, -\xi/q)$ and $(0, 0, \eta_1/p_1)$, lie in the planes (8), while the second plane of (8) also contains the center of geodesic curvature, $(-\eta_1/r_1, 0, 0)$, of the curve $u = \text{const.}$ Hence we have the following theorem.

THEOREM 2. *If the above inversion of C leads to a normal rectilinear congruence, and the curves $v = const.$ are geodesics, $(r \neq 0)$, then two of the focal points of C are collinear with the center of principal curvature of the curve $v = const.$, and the other two are collinear with the centers of principal and geodesic curvature of the curve $u = const.$*

From (6) it is evident that for $p = q_1 = r = 0$, and $R = f(u)$, du and dv are factors of these equations. Hence we have the following theorem.

* R is a function for u alone for (a).

THEOREM 3. *If the above inversion of C leads to a normal rectilinear congruence, and the curves $v=$const. are geodesics, then the circles of C which have an envelope are those which correspond to the lines of curvature on S.*

4. *The Focal Points of L. The Developables of L.* As the vertex of T is displaced along the curves of S which define the developables of L, the displacements δx and δy of the focal points will be zero. Hence from (5), when L is a normal congruence, we have

$$\begin{aligned} &\frac{K^2}{2R^2}\left(\frac{\partial R}{\partial u}du+\frac{\partial R}{\partial v}dv\right)-\xi du-zqdu=0,\\ &\eta_1 dv+\frac{K^2}{2R}(rdu+r_1dv)-zp_1dv=0. \end{aligned} \tag{9}$$

The elimination of dv/du between equations (9) gives us a quadratic in z whose roots determine the focal points of L; the elimination of z gives the equation of the curves on S defining the developables of L.

For (a), that is for $r=0$, $R=f(u)$, the focal points of L as determined by (9) are

$$z_1=\frac{1}{q}\left(\frac{K^2}{2R^2}\frac{\partial R}{\partial u}-\xi\right), \qquad z_2=\frac{1}{p_1}\left(\frac{K^2r_1}{2R}+\eta_1\right). \tag{10}$$

When we put (5) in the second of equations (8) and solve for z we get the value of z_2 in (10). Hence one of the focal points of L is collinear with the two focal points of C determined by the second member of (8). It is readily shown from (5), (8), and (10) that the other focal point of L is not collinear with the other two focal points of C.

From (9) it is readily seen that for (a), du and dv are factors of these equations. Hence we have the following theorem.

THEOREM 4. *If the above inversion of C leads to a normal rectilinear congruence L, and the curves $v=$const. are geodesics, the lines of curvature on S define both the developables of L, and those families of C which have an envelope.*

WESLEYAN UNIVERSITY

ON THE REDUCTION OF A MATRIX TO ITS RATIONAL CANONICAL FORM*

BY M. H. INGRAHAM

Two square $n \times n$ matrices A and B with elements in a field F are similar in F if there exists a non-singular $n \times n$ matrix with elements in F such that $S^{-1}AS = B$. In the study of similarity canonical forms play a fundamental role. The classical canonical form for A is one in which the elementary divisors of $A - \lambda I$ are brought into prominence. In 1926, Dickson published in his *Modern Algebraic Theories* a rational discussion of the problem of similarity in which a rational canonical form based on the invariant factors of $(A - \lambda I)$ was used. Other discussions by Lattés, Krull, Kowalewski, and Menge have been published.

The following seems to be, from the algebraic standpoint, a somewhat more direct discussion than others known to the author. Moreover, in arriving at the well known rational canonical form for a matrix, certain lemmas of interest are developed.

Throughout we consider all elements of matrices and vectors and coefficients of polynomials that enter the discussion to be in a field F. All points of interest are met with if the elements involved are rational.

Consider an $n \times n$ matrix A. If the vectors $\xi_1, \xi_2, \cdots, \xi_p$ are $n \times 1$ matrices, we define $L(\xi_1, \xi_2, \cdots, \xi_p)$ to be the linear set consisting of all vectors of the form $\sum_1^p g_i(A)\xi_i$, where the g's are polynomials.

If L is such a linear set and if g is a polynomial and η an $n \times 1$ vector, we say that $g(A)\eta \equiv 0 \bmod L$, where 0 stands for the zero vector if $g(A)\eta$ is in L.

If $g_1(A)\eta \equiv 0 \bmod L$, and $g_2(A)\eta \equiv 0 \bmod L$, then for every pair of polynomials p_1, p_2,

$$\{p_1(A)g_1(A) + p_2(A)g_2(A)\}\eta \equiv 0 \bmod L.$$

Since the greatest common divisor of g_1 and g_2 is expressible in the form $p_1g_1 + p_2g_2$, one can prove (as in the proof for the existence of a minimum equation for a matrix) that for each

* Presented to the Society, November 25, 1932.

$n \times 1$ vector η and linear set L there exists a polynomial g of lowest degree and leading coefficient unity such that $g(A)\eta \equiv 0 \bmod L$. In fact any other polynomial satisfying this congruence will be divisible by g. The polynomial g is called the minimum function of A relative to (η, L).

If g_1 and g_2 are polynomials, then there exist polynomials f_1, f_2, h_1, and h_2 such that $g_1 = f_1 h_1$, $g_2 = f_2 h_2$, where f_1 and f_2 are relatively prime and $f_1 f_2$ is the least common multiple of g_1 and g_2. This is readily seen, for if the factorizations of g_1 and g_2 into powers of distinct irreducible factors are given by

$$g_1 = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s} q_1^{n_1} q_2^{n_2} \cdots q_t^{n_t}, \quad g_2 = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s} q_1^{u_1} q_2^{u_2} \cdots q_t^{u_t},$$

where $0 < m_i \geqq v_i \geqq 0$ and $0 \leqq n_i \leqq u_i > 0$, then $f_1 = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}$ and $f_2 = q_1^{u_1} q_2^{u_2} \cdots q_t^{u_t}$ are effective. If g_1 is the minimum function of A relative to (η_1, L) and g_2 is the minimum function of A relative to (η_2, L) and if $\eta = h_1(A)\eta_1 + h_2(A)\eta_2$, then g, the minimum function of A relative to (η, L), is $f_1 f_2$, the least common multiple of g_1 and g_2. For we have

$$g(A)\eta = g(A)h_1(A)\eta_1 + g(A)h_2(A)\eta_2 \equiv 0 \bmod L,$$

and hence

$$f_1(A)g(A)h_1(A)\eta_1 + f_1(A)g(A)h_2(A)\eta_2 \equiv 0 \bmod L.$$

But $f_1(A)h_1(A)\eta_1 \equiv 0 \bmod L$, and thus $f_1(A)g(A)h_2(A)\eta_2 \equiv 0 \bmod L$, and $f_1 g h_2$ is a multiple of g_2 and hence g a multiple of f_2. Similarly, g is a multiple of f_1 and since $f_1(A)f_2(A)\eta \equiv 0 \bmod L$, $g = f_1 f_2$. Hence if the minimum function of A relative to (η_1, L) is g_1 and if there exists another vector η_2 for which the minimum function of A relative to (η_2, L) is g_2, where g_2 does not divide g_1, there exists a third vector η_3 such that the minimum function of A relative to (η_3, L) is of higher degree than either g_1 or g_2. As the degree of any such minimum function can not exceed n, there exists a vector η such that g, the minimum function of A relative to (η, L), is divisible by the minimum function of A relative to (η_1, L), where η_1 is any arbitrary vector. Such an η is said to be maximal relative to (A, L) and g is called the minimum function of A relative to L. This process of finding the maximal η corresponds to the finding of the leader of a chain of maximal length at various stages in Dickson's discussion. The

present method, however, gives a mode of construction for this leader.

It is readily seen that if L_1 is contained in L_2, then for any vector η the minimum function of A relative to any vector (η, L_2) is a divisor of the minimum function of A relative to (η, L_1), and that the minimum function of A relative to L_2 is a divisor of the minimum function of A relative to L_1.

Consider a set of vectors $\xi_1, \xi_2, \cdots, \xi_p$. Let $L_0=0, L_1=L(\xi_1)$, $L_j=L(\xi_1, \cdots, \xi_j)$. Let ξ_1 be maximal relative to (A, L_0), ξ_2 maximal relative to (A, L_1), and, in general, ξ_j maximal relative to (A, L_{j-1}). If L_p is not the complete vector space, there exists a vector η not in L_p which is maximal relative to (A, L_p). Let g_i be the minimum function of A relative to L_{i-1}. From the preceding paragraph g_{i-1} is divisible by g_i for each i; hence a set of polynomials k_i exists, such that $g_i=k_ig_{p+1}$, $(i<p+1)$. Since $g_{p+1}(A)\eta\equiv 0 \bmod L_p$, there exist a set of polynomials of f_i such that

$$g_{p+1}(A)\eta = \sum_1^p f_i(A)\xi_i.$$

Moreover, each f_i is divisible by g_{p+1}, for if this were not the case there would be a last number l less than or equal to p for which f_l is not divisible by g_{p+1}. But if j is greater than l, k_lf_j is a multiple of g_l since f_j is assumed to be a multiple of g_{p+1}; hence $k_l(A)f_i(A)\xi_i\equiv 0 \bmod L_{l-1}$ whenever $i\neq l$. Since $k_l(A)g_{p+1}(A)\eta = g_l(A)\eta\equiv 0 \bmod L_{l-1}$, it follows from the last remark that

$$k_l(A)f_l(A)\xi_l \equiv 0 \bmod L_{l-1},$$

but since ξ_l is maximal relative to (A, L_{l-1}), k_lf_l must be a multiple of $g_l=k_lg_{p+1}$ and hence f_l must be a multiple of g_{p+1} in contradiction to our hypothesis. Hence, if $f_i=r_ig_{p+1}$ and if we let

$$\xi_{p+1} = \eta - \sum_1^p r_i(A)\xi_i,$$

it follows that $g_{p+1}(A)\xi_{p+1}=0$ and hence ξ_{p+1}, which is maximal relative to L_p, is such that A has g_{p+1} for its minimum function relative to (ξ_{p+1}, L_0) as well as relative to (ξ_{p+1}, L_p). Hence we may find a set of vectors $\xi_1, \cdots, \xi_l$ with the properties:

(1) Each ξ_i is maximal relative to $L(\xi_1, \cdots, \xi_{i-1})$.

(2) If g_i is the minimum function of A relative to $(\xi_i, L(\xi_1, \cdots, \xi_{i-1}))$, then $g_i(A)\xi_i=0$.

(3) $L(\xi_1, \cdots, \xi_t)$ is the total vector space.

(4) Each ξ_i is different from the zero vector.

Let $g_i(\lambda) = -\sum_1^{n_i-1} b_{ij}\lambda^j + \lambda^{n_i}$. Consider the $n \times n$ matrix, S, whose columns are the linearly independent vectors $\xi_1, A\xi_1, \cdots, A^{n_1-1}\xi_1, \xi_2, A\xi_2, \cdots, A^{n_2-1}\xi_2, \cdots, A^{n_t-1}\xi_t$. If the rows of S^{-1} are the $1 \times n$ vectors $\zeta_1', \cdots, \zeta_n'$, these vectors form with the rows of the transpose of S a biorthogonal system. The columns of AS are

$$A\xi_1, A^2\xi_1, \cdots, A^{n_1-1}\xi_1, \sum_0^{n_1-1} b_{ij}A^j\xi_1, A\xi_2, \cdots, \sum_0^{n_t-1} b_{tj}A^j\xi_t,$$

and

$$\zeta_1'A\xi_1 = 0, \zeta_1'A^2\xi_1 = 0, \cdots, \zeta_1'\sum_0^{n_1-1} b_{ij}A^j\xi_1 = b_{10}, \zeta_1'A\xi_2 = 0, \cdots,$$

$$\zeta_2'A\xi_1 = 1, \zeta_2'A^2\xi_1 = 0, \cdots, \zeta_2'\sum_0^{n_1-1} b_{ij}A^j\xi_1 = b_{11}, \zeta_2'A\xi_2 = 0, \cdots,$$

which are the elements of $S^{-1}AS$, and hence $S^{-1}AS$ is 0 except for $n_i \times n_i$ blocks along the main diagonal of the form

$$\begin{pmatrix} 0 & 0 & 0 \cdots 0 & b_{i0} \\ 1 & 0 & 0 \cdots 0 & b_{i1} \\ 0 & 1 & 0 \cdots 0 & b_{i2} \\ \cdot & \cdot & \cdot \cdots \cdot & \cdot \\ \cdot & \cdot & \cdot \cdots \cdot & \cdot \\ \cdot & \cdot & \cdot \cdots \cdot & \cdot \\ 0 & 0 & 0 \cdots 1 & b_{in_i-1} \end{pmatrix}$$

whose characteristic determinant is $g_i(\lambda)$, where each $g_i(\lambda)$ divides $g_{i-1}(\lambda)$, and hence $S^{-1}AS$ is the transpose of the rational canonical form given by Dickson.

The uniqueness of this canonical form may be established as usual through the invariance of the greatest common divisors of the determinants of the $(n-r)$-rowed minors of $A - \lambda I$, or as follows. Since $g(S^{-1}AS) = S^{-1}g(A)S$, the rank of a polynomial in A and the rank of the same polynomial in any matrix similar to A are identical. Consider two distinct $n \times n$ matrices A_1 and A_2 in canonical form with the characteristic functions of the ith blocks g_{1i} and g_{2i}, respectively; if $g_{1i} = g_{2i}$ for $i < k$ and $g_{1k} \neq g_{2k}$; then if g_{1k} is not divisible by g_{2k}, $g_{2k}(A_2)$ will have rank less than $g_{2k}(A_1)$.

THE UNIVERSITY OF WISCONSIN

ON THE REPRESENTATION OF NUMBERS MODULO m*

BY E. D. RAINVILLE

Dirichlet and Kronecker† extended the notion of primitive root to the case of any composite modulus. The classical Kronecker-Dirichlet theorem may be stated as follows. *Let* $m=2^{\alpha_0}p_1^{\alpha_1}\cdots p_v^{\alpha_v}$, *where the p's are distinct odd primes. Determine* g_k, *a primitive root of* $p_k^{\alpha_k}$, *for* $k=1, 2, \cdots, v$. *Form*

$$\lambda_k = g_k + p_k^{\alpha_k}\beta_k \equiv 1 \mod m/p_k^{\alpha_k},$$

and, if $\alpha_0>1$,

$$\lambda = -1 + 2^{\alpha_0}\beta \equiv 1 \mod m/2^{\alpha_0},$$

$$\lambda_0 = 5 + 2^{\alpha_0}\beta_0 \equiv 1 \mod m/2^{\alpha_0}.$$

Then, for $(n, m)=1$, n *is uniquely represented modulo* m *by*

$$n \equiv \lambda^i\lambda_0^{i_0}\prod_{k=1}^{v}\lambda_k^{i_k} \bmod m,$$

where the exponents are restricted by the inequalities

$$0 \leqq i \leqq 1, \qquad 0 \leqq i_0 \leqq \phi(2^{\alpha_0-1}) - 1, \qquad 0 \leqq i_k \leqq \phi(p_k^{\alpha_k}) - 1.$$

If $\alpha_0 \leqq 1$, λ *and* λ_0 *are not to be formed, hence* $i=i_0=0$ *automatically.*

In the course of another investigation a further extension to the case of general n (dropping the restriction $(n, m)=1$) became necessary. This is the object of the present note.

THEOREM. *Let* $m=2^{\alpha_0}p_1^{\alpha_1}\cdots p_v^{\alpha_v}$ (p*'s distinct odd primes*). *Determine* g_k, *a primitive root*‡ *of* p_k^2, $k=1, 2, \cdots, v$. *Form*

$$\lambda_k = g_k + p_k^{\alpha_k}\beta_k \equiv 1 \mod m/p_k^{\alpha_k}$$

and, if $\alpha_0>1$,

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* Presented to the Society, March 18, 1933.

† Dickson, *History of the Theory of Numbers*, vol. 1, pp. 185, 192.

‡ The root g_k is then also a primitive root of p_k^n, $n>0$ (Dirichlet-Dedekind, *Zahlentheorie*, 4th ed., 1894, p. 334).

$$\lambda = -1 + 2^{\alpha_0}\beta \equiv 1 \mod m/2^{\alpha_0},$$
$$\lambda_0 = 5 + 2^{\alpha_0}\beta_0 \equiv 1 \mod m/2^{\alpha_0}.$$

Then any n is uniquely represented modulo m by

$$\text{(A)} \qquad n \equiv 2^{\sigma_0}\lambda^{i}\lambda_0^{i_0}\prod_{k=1}^{v} p_k^{\sigma_k}\lambda_k^{i_k} \mod m,$$

where

$$0 \leqq \sigma_0 \leqq \alpha_0, \qquad 0 \leqq \sigma_k \leqq \alpha_k, \qquad (k = 1, 2, \cdots, v),$$

and the other exponents are subject to the restrictions that

if $\sigma_0 \geqq \alpha_0 - 1$, *then* $i = i_0 = 0$;
if $0 \leqq \sigma_0 \leqq \alpha_0 - 2$, *then* $0 \leqq i \leqq 1$ *and* $0 \leqq i_0 \leqq \phi(2^{\alpha_0-\sigma_0-1}) - 1$;
if $0 \leqq \sigma_k \leqq \alpha_k$, *then* $0 \leqq i_k \leqq \phi(p_k^{\alpha_k-\sigma_k}) - 1$,

for $k = 1, 2, \cdots, v$.

PROOF. In order to show that all numbers are represented uniquely by (A) we prove (1) that the number of such representations is m, and (2) that no two representations are congruent modulo m.

(1) The number of combinations of exponents σ_0, i, i_0 due to letting σ_0 assume all permissible values is evidently

$$1 + 1 + 2\cdot\sum_{\sigma_0=0}^{\alpha_0-2}\phi(2^{\alpha_0-\sigma_0-1}) = 1 + \phi(2) + \sum_{\sigma_0=0}^{\alpha_0-2}\phi(2^{\alpha_0-\sigma_0})$$
$$= \sum_{\sigma_0=0}^{\alpha_0}\phi(2^{\alpha_0-\sigma_0}) = 2^{\alpha_0}.$$

Similarly, for any $k = 1, 2, \cdots, v$, the number of combinations of exponents σ_k, α_k due to letting σ_k assume all permissible values is

$$\sum_{\sigma_k=0}^{\alpha_k}\phi(p_k^{\alpha_k-\sigma_k}) = p_k^{\alpha_k}.$$

Hence, combining these results, we have for T, the total number of representations,

$$T = 2^{\alpha_0}p_1^{\alpha_1}\cdots p_v^{\alpha_v} = m.$$

(2) The uniqueness is made to depend upon the Kronecker-Dirichlet theorem in the following manner. Suppose, with the restrictions of our theorem, that

$$2^{\sigma_0'}\lambda^{i}\lambda_0^{i_0}\prod_{k=1}^{v}p_k^{\sigma_k}\lambda_k^{i_k} \equiv 2^{\sigma_0'}\lambda^{i'}\lambda_0^{i_0'}\prod_{k=1}^{v}p_k^{\sigma_k'}\lambda_k^{i_k'} \mod m.$$

Then, since λ, λ_0 and λ_k are relatively prime to m,

$$\sigma_0=\sigma_0', \quad \sigma_k=\sigma_k', \quad (k=1, 2, \cdots, v),$$

and we have

$$\text{(B)} \quad \lambda^{i}\lambda_0^{i_0}\prod_{k=1}^{v}\lambda_k^{i_k} \equiv \lambda^{i'}\lambda_0^{i_0'}\prod_{k=1}^{v}\lambda_k^{i_k'} \bmod 2^{\alpha_0-\sigma_0}\prod_{k=1}^{v}p_k^{\alpha_k-\sigma_k}.$$

Since λ_k is a primitive root of p_k^2, it is a primitive root of $p_k^{\alpha_k-\sigma_k}$. From the restrictions of the theorem, we conclude that $0\leqq i_k,\ i_k'\leqq\phi(p_k^{\alpha_k-\sigma_k})-1$. Further, $0\leqq i_0,\ i_0'\leqq\phi(2^{\alpha_0-\sigma_0-1})-1$, if only $\alpha_0-\sigma_0>1$. Again, if $\alpha_0-\sigma_0>1$, we know that $0\leqq i, i'\leqq 1$.

Thus all conditions of the Kronecker-Dirichlet theorem are satisfied in (B) for the modulus

$$2^{\alpha_0-\sigma_0}\prod_{k=1}^{v}p_k^{\alpha_k-\sigma_k} = (m/2^{\sigma_0})\prod_{k=1}^{v}p_k^{\sigma_k},$$

and the representation $\lambda^{i}\lambda_0^{i_0}\prod_{k=1}^{v}\lambda_k^{i_k}$ is a unique representation modulo $(m/2^{\sigma_0})\prod_{k=1}^{v}p_k^{\sigma_k}$, and, a fortiori, modulo m. Therefore, the representation (A) is unique modulo m.

THE UNIVERSITY OF COLORADO

NOTE ON A PREVIOUS PAPER

BY B. F. KIMBALL

Dr. Hillel Poritsky has called my attention to the fact that the second theorem in my previous paper* can be treated more completely by the method of integral equations. No restriction concerning the existence of a first derivative of the force function $f(x)$ need be made, under the assumption of a constant period. We wrote

$$T(a) = 2 \cdot 2^{1/2} \int_0^a \frac{dx}{(F(a) - F(x))^{1/2}},$$

where T was the period and a the amplitude, with $F(a) = \int_0^a f(x)dx$. Under the transformation $z = F(x)$, $h = F(a)$, $u(z) = 1/f(x)$, this becomes

$$T(a) = 2 \cdot 2^{1/2} \int_0^h \frac{u(z)dz}{(h - z)^{1/2}} = 2 \cdot 2^{1/2}\phi(h).$$

Under the hypothesis of constant period (and $f(x)$ non-vanishing for x greater than zero) we have $\phi'(h) \equiv 0$, and $\phi(h) \equiv \phi(0)$. The solution of the integral equation

$$\phi(h) = \int_0^h \frac{u(z)dz}{(h - z)^{1/2}}$$

gives

$$u(h) = \frac{1}{\pi}\left[\frac{\phi(0)}{h^{1/2}} + \int_0^h \frac{\phi'(z)dz}{(h-z)^{1/2}}\right],$$

which leads to

$$f(x) \equiv (4\pi^2/T^2)x.$$

SCHENECTADY, NEW YORK

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* *Three theorems applicable to vibration theory*, this Bulletin, vol. 38 (1932), pp. 718–723.

ON COMPLETE INDEPENDENCE OF CERTAIN SETS OF POSTULATES FOR FIELDS*

BY N. H. McCOY

1. *Introduction.* Definitions of a field by sets of independent postulates have been given by a number of authors,† but in most cases the question of complete independence in the sense of E. H. Moore‡ has not been considered. The primary purpose of this note is to present a method which may be used to advantage in the study of complete independence of any set of field postulates except those which contain a postulate specifying the cardinal number of the system as a given finite number. The method will be presented by applying it to establish the complete independence of a set of seven postulates, essentially the Definition 1 of Huntington.§ In §4 a very short proof of the complete independence of Hurwitz's‖ five postulates for fields is also given. This set was previously shown by Bernstein¶ to be completely independent.

2. *The Seven Postulates.* Let K be a set of undefined elements $a, b, c, \cdots$. We assume the existence of two uniquely defined binary operations $\oplus$ and $\odot$ on the elements of K as well as an *equality* $\ominus$ which is reflexive, symmetric, and transitive.** The system $[K, \oplus, \odot, \ominus]$ will be a field if it has the following seven properties:††

* Presented to the Society, December 29, 1932.

† For a list of references see a recent article by W. A. Hurwitz, Annals of Mathematics, (2), vol. 33 (1932), pp. 403–405. See also an abstract by C. R. Worth, this Bulletin, vol. 38 (1932), p. 350.

‡ E. H. Moore, *Introduction to a Form of General Analysis*, New Haven Mathematical Colloquium, Yale University Press, p. 82.

§ E. V. Huntington, Transactions of this Society, vol. 4 (1903), pp. 31–37.

‖ W. A. Hurwitz, Annals of Mathematics, (2), vol. 15 (1913), pp. 93–100.

¶ B. A. Bernstein, Annals of Mathematics, (2), vol. 23 (1921–22), pp. 313–316.

** See, for example, O. Haupt, *Einführung in die Algebra*, vol. 1, p. 2. For non-equality, we shall say "does not $\ominus$."

†† E. V. Huntington, loc. cit., p. 34. This set of properties is the Definition 1 of Huntington with the exception that $P6$ is made slightly stronger in statement. This set was later modified by Huntington; see Transactions of this Society, vol. 6 (1905), pp. 181–193.

*P*1. If a, b, c, $a \oplus b$, $b \oplus c$, and $a \oplus b(\oplus c)$ belong to K, then $(a \oplus b) \oplus c \ominus a \oplus (b \oplus c)$.

*P*2. If a, b, and $b \oplus a$ belong to K, then $a \oplus b \ominus b \oplus a$.

*P*3. If a and b belong to K, there is an element x of K such that $a \oplus x \ominus b$.

*P*4. If a, b, c, $a \odot b$, $b \odot c$, and $a \odot (b \odot c)$ belong to K, then $(a \odot b) \odot c \ominus a \odot (b \odot c)$.

*P*5. If a, b, $b \odot a$ belong to K, then $a \odot b \ominus b \odot a$.

*P*6. If a and b belong to K and $a \oplus a$ does not $\ominus a$, there is an element y of K such that $a \odot y \ominus b$.

*P*7. If a, b, c, $b \oplus c$, $a \odot b$, $a \odot c$, and $(a \odot b) \oplus (a \odot c)$ belong to K, then $a \odot (b \oplus c) \ominus (a \odot b) \oplus (a \odot c)$.

We shall now show that this set of properties is completely independent.

3. *Proof of Complete Independence.* To establish the complete independence of the above seven properties we need to show the existence of 2^7 systems having each of the possible characters with respect to these properties. Each of the systems used below is an infinite system and each one is actually closed with respect to the operations $\oplus$ and $\odot$.

Let S_i denote a system $[K_i, (+)_i, (\cdot)_i, (=)_i]$, and let $a_i, b_i, \cdots$ denote elements of K_i, $(i = 1, 2, \cdots)$. We now define the *multiple system* $(S_i, S_j, \cdots, S_k)$, formed from the *component systems* $S_i, S_j, \cdots, S_k$, and consisting of all elements of the form $(a_i, a_j, \cdots, a_k)$. The symbols $\oplus$, $\odot$, $\ominus$ for this multiple system will be denoted by $[+]$, $[\cdot]$ and $[=]$ respectively, and are defined as follows:

$$(a_i, a_j, \cdots, a_k)[+](b_i, b_j, \cdots, b_k)$$

is the element

$$(a_i(+)_i b_i, a_j(+)_j b_j, \cdots, a_k(+)_k b_k);$$

$$(a_i, a_j, \cdots, a_k)[\cdot](b_i, b_j, \cdots, b_k)$$

is the element

$$(a_i(\cdot)_i b_i, a_j(\cdot)_j b_j, \cdots, a_k(\cdot)_k b_k);$$

$$(a_i, a_j, \cdots, a_k)[=](b_i, b_j, \cdots, b_k)$$

means

$$a_i(=)_i b_i, a_j(=)_j b_j, \cdots, a_k(=)_k b_k.$$

The following theorems are now evident.

THEOREM 1. *If any one of the component systems* $S_i, S_j, \cdots, S_k$ *does not have property* Pr, *the multiple system* $(S_i, S_j, \cdots, S_k)$ *does not have property* Pr, $(r = 1, 2, \cdots, 7)$.

THEOREM 2. *If each of the component systems* $S_i, S_j, \cdots, S_k$ *has property* Pr, $(r \neq 6)$, *the multiple system* $(S_i, S_j, \cdots, S_k)$ *also has this property.*

In order to state the corresponding theorem for $P6$, we make use of the following two auxiliary properties:

I. If a belongs to K, then $a \oplus a \ominus a$.

II. If a and b belong to K, there exists an element y of K such that $a \odot y \ominus b$.

We now prove the following theorem.

THEOREM 3. *If each of the component systems* $S_i, S_j, \cdots, S_k$ *has property* $P6$, *the system* $(S_i, S_j, \cdots, S_k)$ *also has this property if* (1) *each of the component systems has property* I, *or* (2) *each of the component systems has property* II, *or* (3) *some one of the component systems has neither of the properties* I *and* II *and the remaining ones have both these properties.*

It is readily verified that the multiple system $(S_i, S_j, \cdots, S_k)$ has property $P6$ in cases (1) and (2). In case (3) suppose that S_i has neither property I nor property II. If

$$(a_i, a_j, \cdots, a_k)[\cdot](y_i, y_j, \cdots, y_k)[=](b_i, b_j, \cdots, b_k)$$

admits no solution for $(y_i, y_j, \cdots, y_k)$, then the equation $a_i(\cdot)_i y_i (=)_i b_i$ admits no solution in K_i, and since S_i has property $P6$, we must have $a_i(+)_i a_i (=)_i a_i$. Hence

$$(a_i, a_j, \cdots, a_k)[+](a_i, a_j, \cdots, a_k)[=](a_i, a_j, \cdots, a_k),$$

and the multiple system thus has property $P6$.

The table defines ten systems which are fundamental in our proof of the complete independence of the seven properties. Henceforth, as in the table, S_i will denote a system which has each of the seven properties with the exception of Pi, S_{ij} a system which has neither of the properties Pi and Pj but each of the other properties. Thus systems $S_1, S_2, \cdots, S_7$ show the

ordinary independence of the set.* The last column of the table gives the character of each system with respect to properties I and II, and is for convenience in applying Theorem 3.

By means of Theorems 1 and 2 we can construct at once from $S_1, S_2, \cdots, S_7$ a system not having property $P6$ but having any given combination of the remaining properties. For if we wish, for example, a system having each of the properties with the exception of $P1$, $P5$, and $P6$, we have merely to use the system (S_1, S_5, S_6). This disposes of the 64 systems which do not have property $P6$ and henceforth we shall only consider systems which have this property.

TABLE OF FUNDAMENTAL SYSTEMS

System	K	$a \oplus b$	$a \odot b$	$a \ominus b$	I	II
S_1	Positive reals	$+(ab)^{1/2}$	ab	$a=b$	+	+
S_2	Positive rationals	b	ab	$a=b$	+	+
S_3	Positive rationals	$a+b$	ab	$a=b$	−	+
S_4	Number pairs $a=(a_1, a_2)$, with a_1, a_2 rational	(a_1+b_1, a_2+b_2)	$(a_1b_1-a_2b_2, -a_1b_2-a_2b_1)$	$\begin{cases} a_1=b_1 \\ a_2=b_2 \end{cases}$	−	−
S_5	Rationals	$a+b$	b	$a=b$	−	+
S_6	Rationals	$a+b$	0	$a=b$	−	−
S_7	Rationals	$a+b$	$a+b$	$a=b$	−	+
S_{34}	Same as in S_4	$(0, 0)$	Same as in S_4	Same as in S_4	−	−
S_{45}	Rationals	$a+b$	$-b$	$a=b$	−	+
S_{47}	Rationals	$a+b$	$-(a+b)$	$a=b$	−	+

Now each of the systems S_1, S_2, S_3, S_5, S_7 has property II, and by Theorem 3, property $P6$ will be present in any multiple system constructed from them. We can therefore, as above, construct a system which has properties $P6$ and $P4$ and any combination of the remaining properties. Hence in order to establish complete independence we only need to exhibit 32 systems having property $P6$ and not having property $P4$ but admitting each possible character with respect to the remaining five properties.

* The systems S_1, S_2, S_3, S_4 were used by Huntington (loc. cit.), the last of these being due to Dickson. Systems S_5, S_6, S_7 were used by Bernstein (loc. cit.).

If S denotes any one of these 32 remaining systems and if it has property $P1$, then the system (S_1, S) will be a system having the same character as S with the exception that property $P1$ is absent. This is true by Theorem 3 as S_1 has both properties I and II. A similar statement holds for property $P2$. Thus the existence of the remaining 32 systems will have been established when we exhibit the eight systems having properties $P1$, $P2$, and $P6$ and not having property $P4$ but having each combination of the other properties. It may be verified that the following eight systems satisfy these requirements: S_4, S_{34}, S_{45}, S_{47}, (S_3, S_{45}), (S_3, S_{47}), (S_{45}, S_7), and (S_3, S_5, S_{47}). This completes the proof of the complete independence of the seven properties.

4. *Complete Independence of Hurwitz's Five Postulates.* Hurwitz* has given a definition of a field by means of a set of five postulates, which are our $P3$, $P6$, $P7$, and the following two:

$P12$. If $a, b, c, a \oplus b, c \oplus b$, and $a \oplus (c \oplus b)$ belong to K, then $(a \oplus b) \oplus c \ominus a \oplus (c \oplus b)$.

$P45$. If $a, b, c, a \odot b, c \odot b$, and $a \odot (c \odot b)$ belong to K, then $(a \odot b) \odot c \ominus a \odot (c \odot b)$.

It is seen that Theorems 1, 2, and 3 hold also for this set of five properties. Ordinary independence is shown by the systems S_3, S_6, S_7, S_1, and S_5. But with the exception of S_6 each of these systems has property II. Hence it is easily verified that any multiple system formed from these will have no one of the properties which is lacking in any of the component systems and will have each property which is present in each component system. From the above five systems we can therefore form at once a multiple system having any desired combination of the five properties. Thus the set is completely independent. Another proof of this fact has been given by Bernstein.†

SMITH COLLEGE

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* W. A. Hurwitz, Annals of Mathematics, (2), vol. 15 (1913), pp. 93–100.

† Loc. cit. See also this Bulletin, vol. 28 (1922), pp. 397–399.

ON THE NUMBER OF $(q+1)$-SECANT S_{q-1}'S OF A CERTAIN V_k^n IN AN $S_{qk+q+k-1}$

BY B. C. WONG

In this note we are concerned only with those k-dimensional non-developable varieties which are rational loci each of ∞^1 $(k-1)$-spaces. By a rational locus of ∞^1 $(k-1)$-spaces we mean one whose $(k-1)$-spaces can be put in a one-to-one correspondence with the points of a straight line. Let such a locus or variety, V_k^n, of order n be given in an S_r. Now in S_r there are $\infty^{q(r-q+1)}$ $(q-1)$-spaces. For a $(q-1)$-space to meet V_k^n $q+1$ times is equivalent to $(q+1)(r-q-k+1)$ simple conditions. In order that the number, N, of $(q-1)$-spaces $(q+1)$-secant to V_k^n, that is, having $q+1$ points of simple incidence with V_k^n, be finite, we must have $(q+1)(r-q-k+1)=q(r-q+1)$ or $r=qk+q+k-1$. It is our purpose to determine this number N of $(q+1)$-secant S_{q-1}'s of V_k^n in $S_{qk+q+k-1}$.

For this purpose we find it convenient to consider the V_k^n in question as the projection of a $V_k'^n$ in a higher space $S_{r'}$. This $V_k'^n$ may always be regarded as the locus of ∞^1 $(k-1)$-spaces joining corresponding points of k rational, projectively related curves C^{n_1}, C^{n_2}, $\cdots$, C^{n_k} of respective orders $n_1, n_2, \cdots, n_k$, where $n_1+n_2+\cdots+n_k=n$. The $S_{r'}$ containing $V_k'^n$ must be such that $r' \leqq n+k-1$. If $r'=n+k-1$, $V_k'^n$ is said to be normal in S_{n+k-1}. It is only necessary to consider this normal $V_k'^n$.

Let the k curves be given parametrically by

$$C^{n_1} \quad x_0 : x_1 : \cdots : x_{n_1} = t^{n_1} : t^{n_1-1} : \cdots : 1,$$
$$x_{n_1+1} = x_{n_1+2} = \cdots = x_{n+k-1} = 0;$$

$$C^{n_2} \quad x_0 = x_1 = \cdots = x_{n_1} = 0,$$
$$x_{n_1+1} : x_{n_1+2} : \cdots : x_{n_1+n_2+1} = t^{n_2} : t^{n_2-1} : \cdots : 1,$$
$$x_{n_1+n_2+2} = x_{n_1+n_2+3} = \cdots = x_{n+k-1} = 0;$$

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$$C^{n_k} \quad x_0 = x_1 = \cdots = x_{n-n_k+k-2} = 0,$$
$$x_{n-n_k+k-1} : x_{n-n_k+k} : \cdots : x_{n+k-1} = t^{n_k} : t^{n_k-1} : \cdots : 1.$$

Then a general point of $V_k'^n$ has the coordinates

$$(\lambda_1 t^{n_1}:\lambda_1 t^{n_1-1}:\cdots:\lambda_1:\lambda_2 t^{n_2}:\lambda_2 t^{n_2-1}:\cdots:\lambda_2:\cdots:$$
$$\cdots:\lambda_k t^{n_k}:\lambda_k t^{n_k-1}:\cdots:\lambda_k).$$

Now let t take on $q+1$ values, say $t_0, t_1, \cdots, t_q$, and we have $q+1$ points on $V_k'^n$ determining an S_q. The parametric equations of this S_q are, the parameters being the l's,

$$x_{n-n_k+k-1+j_k} = \lambda_k \sum_{i=0}^{q} (l_i t_i^{n_k-j_k}),$$
$$[h = 1, 2, \cdots, k;\ j_h = 1, 2, \cdots, n_h].$$

If we now eliminate the t's, l's, and λ's from the above equations of S_q, we obtain

$$\left\| \begin{matrix} x_0 & x_1 & \cdots & x_{n_1-q-1} & x_{n_1+1} & x_{n_1+2} & \cdots & x_{n_1+n_2-q} \\ x_1 & x_2 & \cdots & x_{n_1-q} & x_{n_1+2} & x_{n_1+3} & \cdots & x_{n_1+n_2-q+1} \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ x_{q+1} & x_{q+2} & \cdots & x_{n_1} & x_{n_1+q+2} & x_{n_1+q+3} & \cdots & x_{n_1+n_2+1} \\ \cdots & \cdots & & x_{n-n_k+k-1} & x_{n-n_k+k} & & \cdots & x_{n+k-q-2} \\ \cdots & \cdots & & x_{n-n_k+k} & x_{n-n_k+k+1} & & \cdots & x_{n+k-q-1} \\ & & & \cdot & \cdot & & & \cdot \\ & & & \cdot & \cdot & & & \cdot \\ \cdots & \cdots & & x_{n-n_2+k+q} & x_{n-n_k+k+q+1} & \cdots & & x_{n+k-1} \end{matrix} \right\| = 0.$$

These are the equations of a $(qk+q+k)$-dimensional variety V_{qk+q+k}^{M} of order M. This variety is the locus of the $\infty^{k(q+1)}$ q-spaces each meeting $V_k'^n$ $q+1$ times. To determine M, notice that the matrix in the left-hand member of the above equality consists of $n-qk$ columns and $q+2$ rows. Applying the rule given by Salmon* for the determination of the order of a restricted system of equations, we find that the order of V_{qk+q+k}^{M} is

$$M = \binom{n-qk}{q+1}.$$

Since V_{qk+q+k}^{M} is in S_{n+k-1}, an $S_{n-qk-q-1}$ of S_{n+k-1} meets it in

* *Modern Higher Algebra*, 4th ed., Lesson 19.

M points. Now let both $V_k'^n$ and V_{qk+q+k}^M be projected from $S_{n-qk-q-1}$ upon an S_{qk+q+k}. The projection of the former is a $V_k''^n$ and that of the latter is a system of $\infty^{k(q+1)}$ q-spaces. Each of these q-spaces is $(q+1)$-secant to $V_k''^n$ and M of them pass through a given point P. If we now project $V_k''^n$ from P upon an $S_{qk+q+k-1}$ of S_{qk+q+k}, we obtain for projection the V_k^n the number N of whose $(q+1)$-secant $(q-1)$-spaces we wish to find. The $(q+1)$-secant S_{q-1}'s of V_k^n are the $(q-1)$-spaces in which $S_{qk+q+k-1}$ intersects the $(q+1)$-secant S_q's of $V_k''^n$ passing through P. Hence the number N we are seeking is equal to M, that is,

$$N = \binom{n-qk}{q+1}.$$

Thus, for $k=1$, we have a rational curve C^n in S_{2q} having $\binom{n-q}{q+1}$ $(q+1)$-secant S_{q-1}'s. If $q=1$, we have the familiar case of a rational plane curve of order n with $(n-1)(n-2)/2$ double points. If $q=2$, we have the case which is also familiar of a rational 4-space curve having $(n-2)(n-3)(n-4)/6$ trisecant lines.

Let $k=2$ and we have a rational ruled surface F^n of order n in S_{3q+1} with $\binom{n-2q}{q+1}$ $(q+1)$-secant S_{q-1}'s. Thus, a rational F^n in S_4 has $(n-2)(n-3)/2$ improper double points; an F^n in S_7 has $(n-4)(n-5)(n-6)/6$ trisecant lines.

If we put $k=3$ and then $q=1, 2, 3, \cdots$, successively, we find, by what precedes, that a rational planed variety V_3^n of order n in S_6, S_{10}, $S_{14}, \cdots$, has, respectively, $(n-3)$ $(n-4)/2$ improper double points, $(n-6)(n-7)(n-8)/6$ trisecant lines, $(n-9)(n-10)(n-11)(n-12)/24$ quadrisecant planes, $\cdots$.

THE UNIVERSITY OF CALIFORNIA

ON THE CONVERGENCE OF FOURIER SERIES

BY WILLIAM RANDELS

The purpose of this note is to show that a criterion for the convergence of Fourier series, given by Tonelli,* is contained in the Lebesgue criterion.

The conditions of the Tonelli criterion are†

(1) $$\phi(t) = o(1) \text{ as } t \to 0;$$

(2) $\phi(t)$ is absolutely continuous in the interval (ϵ, π), $\epsilon > 0$, or

$$\phi(t) = \int_{t_1}^{t} \phi'(\tau)d\tau, \text{ for } t > 0;$$

(3) meas. $\underline{\lim}_{t \to 0} t\phi'(t) \geqq 0$.

Here by meas. $\underline{\lim}$ we mean that in calculating the limit we are permitted to leave out of consideration sets of measure zero. We have to show that, if the above conditions are satisfied, the conditions of the Lebesgue criterion,

(4) $$\Phi(t) = \int_0^t |\phi(\tau)| d\tau = o(t),$$

(5) $$\int_\delta^a |\phi(t+\delta) - \phi(t)| \frac{dt}{t} = o(1), \text{ as } \delta \to 0, \delta > 0,$$

are also satisfied.

Condition (4) is an obvious consequence of (1). Let $\phi'(t) = \phi_1'(t) + \phi_2'(t)$, where

$$\phi_1'(t) = \begin{cases} \phi'(t), t \subset E \equiv E(\phi'(t) \geqq 0), \\ 0, t \subset C(E), \end{cases} \qquad \phi_2'(t) = \begin{cases} 0, t \subset E, \\ \phi'(t), t \subset C(E). \end{cases}$$

Then $|\phi'(t)| = \phi_1'(t) - \phi_2'(t) = \phi'(t) - 2\phi_2'(t)$. Since $\phi_2'(t) < 0$, condition (3) implies that we have meas. $\underline{\lim}_{t \to 0} t\phi_2'(t) = 0$, or $\phi_2'(t) = o(1/t)$, except for a set of measure zero. Since we know that $\delta > 0$ and $\phi(t) = \int_{t_1}^{t} \phi'(\tau)d\tau$ if $t > 0$,

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* L. Tonelli, *Serie Trigonometriche*, 1928, p. 291.

† $\phi(t)$ is defined in the usual way by $\phi(t) = f(x+t) + f(x-t) - 2f(x)$.

$$\int_\delta^a |\phi(t+\delta)-\phi(t)|\frac{dt}{t}$$

$$= \int_\delta^a \frac{dt}{t}\left|\int_t^{t+\delta}\phi'(\tau)d\tau\right| \leqq \int_\delta^a \frac{dt}{t}\int_t^{t+\delta}|\phi'(\tau)|\,d\tau = I.$$

By the Fubini theorem the order of integration may be interchanged, and as a result,

$$I = \int_\delta^{2\delta}|\phi'(\tau)|\,d\tau\int_\delta^\tau\frac{dt}{t} + \int_{2\delta}^a|\phi'(\tau)|\,d\tau\int_{\tau-\delta}^\tau\frac{dt}{t}$$

$$+\int_a^{a+\delta}|\phi'(\tau)|\,d\tau\int_{\tau-\delta}^a\frac{dt}{t}$$

$$= \int_\delta^{2\delta}|\phi'(\tau)|\log\frac{\tau}{\delta}d\tau + \int_{2\delta}^a|\phi'(\tau)|\log\frac{\tau}{\tau-\delta}d\tau$$

$$+\int_a^{a+\delta}|\phi'(\tau)|\log\frac{a}{\tau-\delta}d\tau = I_1+I_2+I_3.$$

We assume now that a is taken sufficiently small; then

$$I_1 \leqq \log 2\int_\delta^{2\delta}|\phi'(\tau)|\,d\tau = \log 2\left(\int_\delta^{2\delta}\phi'(\tau)d\tau - 2\int_\delta^{2\delta}\phi_2'(\tau)d\tau\right)$$

$$= o\left(1+\int_\delta^{2\delta}\frac{d\tau}{\tau}\right) = o(1),$$

$$I_2 = \int_{2\delta}^a\phi'(\tau)\log\frac{\tau}{\tau-\delta}d\tau - 2\int_{2\delta}^a\phi_2'(\tau)\log\frac{\tau}{\tau-\delta}d\tau = I_2'+2I_2'',$$

$$I_2' = \phi(\tau)\log\frac{\tau}{\tau-\delta}\Big]_{2\delta}^a + \delta\int_{2\delta}^a\phi(\tau)\frac{d\tau}{\tau(\tau-\delta)}$$

$$= o\left(1+\delta\int_\delta^a\frac{d\tau}{\tau^2}\right) = o(1),$$

$$I_2'' = -\int_{2\delta}^a\phi_2'(\tau)\log\frac{\tau}{\tau-\delta}d\tau = \int_{2\delta}^a\phi_2'(\tau)\log\frac{\tau-\delta}{\tau}d\tau$$

$$= o\left(\int_{2\delta}^a\left|\log\left(1-\frac{\delta}{\tau}\right)\right|\frac{d\tau}{\tau}\right),$$

$$\int_{2\delta}^{a} \left|\log\left(1-\frac{\delta}{\tau}\right)\right| \frac{d\tau}{\tau} = -\int_{1/2}^{\delta/a} |\log(1-\tau)| \frac{d\tau}{\tau}$$

$$= \int_{\delta/a}^{1/2} |\log(1-\tau)| \frac{d\tau}{\tau} = o(1).$$

Hence $I_2 = o(1)$. Finally it is obvious that $I_3 = o(1)$ as $\delta \to 0$. On combining these facts, we obtain the desired result:

$$\int_{\delta}^{a} |\phi(t+\delta) - \phi(t)| \frac{dt}{t} = o(1).$$

BROWN UNIVERSITY

A NOTE ON COMPACTNESS*

BY E. H. HANSON

The purpose of this note is to deduce the conditions for compactness† of a set of measurable functions from the general criterion for compactness in complete metric spaces.‡ This is the procedure that suggests itself immediately and it suceeds without any difficulty. The general criterion referred to asserts that a set S of elements of a complete metric space is compact if and only if, for every positive ϵ, S is inclosable in a finite number of spheres§ of radius ϵ; and the validity of this the reader may easily verify for himself. Fréchet‖ has obtained the result for

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* Presented to the Society, February 25, 1933. I wish to express my gratitude to Professor Henry Blumberg for suggesting the idea of this note and for helpful criticism during its preparation.

† A set S of elements of a space is *compact* if every infinite subset of S has at least one limit point in the space.

‡ A space M is said to be *metric* if there exists a positive or zero function $d(e_1, e_2)$ of pairs of elements of M satisfying the conditions: (1) $d(e_1, e_2) = d(e_2, e_1)$, (2) $d(e_1, e_2) = 0$ is equivalent to $e_1 = e_2$; (3) $d(e_1, e_3) \leq d(e_1, e_2) + d(e_2, e_3)$. A metric space is *complete* if $\lim_{m,n\to\infty} d(e_m, e_n) = 0$ implies the existence of an element e such that $\lim_{n\to\infty} d(e_n, e) = 0$.

§ A *sphere* with center c and radius r is by definition the set of elements e of M such that $d(e, c) < r$.

‖ *Sur les ensembles compacts de fonctions mesurables*, Fundamenta Mathematicae, vol. 9, p. 25.

measurable functions but without making the deduction in the direct manner made possible by the use of the general criterion; the deduction given here, it would seem, is the simplest available.

To apply the above criterion to the case of measurable functions we must define the distance or écart of two measurable functions. The idea leading to this definition is that the distance between two measurable functions will be regarded as small if the functions differ by a small amount except in a set of small measure. The distance $d(f, \phi)$ of two measurable functions $f(x)$ and $\phi(x)$ defined on a bounded measurable set E will accordingly be defined as the lower boundary of all positive numbers ω such that $|f(x)-\phi(x)| < \omega$ except at the points of a set of measure $< \omega$. Convergence in terms of this distance is evidently equivalent to convergence in measure. That the set of all measurable functions defined on E constitutes a metric space follows at once if we agree to regard as identical two measurable functions which differ only on a set of measure zero. It is also easy to verify that this space is complete.* It follows that the above general criterion holds for this space. This is the result desired except that what is left to do is to mold it over into another form; and here the argument is at once suggested by the corresponding argument for continuous functions and by the fact that every measurable function is equal to a continuous function with an arbitrarily small error, except in a set of arbitrarily small measure. For continuous functions the resulting theorem is: A necessary and sufficient condition that a set of continuous functions defined on a bounded, closed set be compact† is that the set of functions be equi-bounded (uniformly bounded) and equi-continuous.‡

Let S be any set of measurable functions defined on a bounded measurable set E and inclosable, for every $\epsilon > 0$, in a finite number of spheres of radius ϵ. For a given ϵ denote the spheres by

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* Or we may note that this is merely a restatement of a well known property of convergence in measure. See Hobson, *Theory of Functions of a Real Variable*, vol. II, p. 244.

† For continuous functions $d(f_1, f_2) = \text{maximum } |f_1(x)-f_2(x)|$.

‡ A set of continuous functions is said to be *equi-continuous* if, for every positive ϵ, there exists a positive number δ such that $|x_1-x_2| < \delta$ implies for every function $f(x)$ of the set, that $|f(x_1)-f(x_2)| < \epsilon$.

H_i and their centers by $f_i(x)$, $[i=1, 2, \cdots, n]$. Since $f_\nu(x)$ is measurable it can be approximated by a continuous function $c_\nu(x)$ with an error $<\epsilon$ except in a set of E_ν of measure $<\epsilon$. Since $f_0(x)$ may be extended to a measurable function defined in a closed interval I containing E, and since $c_\nu(x)$ may be defined on I, there is no restriction in assuming that $c_\nu(x)$ is uniformly continuous on E. If $f(x)$ is in the sphere with center $f_\nu(x)$, we have $d(f_\nu, f)<\epsilon$ and since $d(c_\nu, f_\nu)<\epsilon$ we have $d(f, c_\nu)<2\epsilon$, that is, $f(x)$ differs from $c_\nu(x)$ by less than 2ϵ except on a set of measure $<2\epsilon$. Since the $c_i(x)$ are continuous and finite in number it follows that they are equi-bounded and equi-continuous. Therefore S is *almost equi-bounded* and *almost equi-continuous* in the sense that, for every $\epsilon>0$, there exist two positive numbers M and δ such that to every function $f(x)$ of S there corresponds a set E_f of measure $<\epsilon$ such that, except on E_f, $|f(x)|<M$ and such that, for every x_1, x_2 belonging to $E-E_f$, $|x_1-x_2|<\delta$ implies $|f(x_1)-f(x_2)|<\epsilon$.

What now remains is to verify the converse, namely, that, if S is any almost equi-bounded and almost equi-continuous set of measurable functions defined on a bounded measurable set E, then, for every $\epsilon>0$, S is inclosable in a finite number of spheres of radius ϵ. Let, then, ϵ be given. Since E lies in a finite interval and since S is almost equi-bounded and almost equi-continuous, it is possible to determine a rectangle R and a positive number δ such that with every function $f(x)$ of S there is associated a set E_f of measure $<\epsilon$ such that, on $E-E_f$, $f(x)$ lies entirely in the interior of R and such that, x_1, x_2 being any two points of $E-E_f$, $|x_1-x_2|<\delta$ implies $|f(x_1)-f(x_2)|<\epsilon$. Divide R into a finite number of vertical strips of width $<\delta$ by means of vertical lines having abscissas $x_0, x_1, \cdots, x_r$ and also into a finite number of horizontal strips of width $<\epsilon$ by lines having the ordinates $y_0, y_1, \cdots, y_s$. Let $f(x)$ now denote a particular function of S and consider the corresponding set $E-E_f$. The measure of this set is $>m(E)-\epsilon$. We may accordingly determine a closed subset F of $E-E_f$ which is also of measure $>m(E)-\epsilon$. We now define a measurable function $\phi(x)$ on the set E as follows. If there are no points of F in the interval $x_k \leqq x < x_{k+1}$, we define $\phi(x)=0$ for every point of E in this interval. If there are points of F in $x_k \leqq x < x_{k+1}$, let ξ be the first. The point $[\xi, f(\xi)]$ is in R. Suppose $y_l \leqq f(\xi) < y_{l+1}$. We define

$\phi(x)=y_i$ for every point of E in $x_k \leqq x < x_{k+1}$. Then, at every point of F, $|f(x)-\phi(x)| < \epsilon$, and $m(E-F) < \epsilon$. Hence $d(f, \phi) < \epsilon$. But the number of such functions $\phi(x)$ is finite, and S is inclosable in the above sense. We have shown that, for the space of measurable functions, the general criterion for compactness in complete metric spaces is equivalent to the following theorem.

THEOREM OF FRÉCHET. *A necessary and sufficient condition that a set S of measurable functions defined on a bounded measurable set E be compact is that the functions of S be almost equibounded and almost equi-continuous.*

The difficulty of generalizing the above results to general functions so as to obtain useful results is shown by an example due to Sierpiński* from which it is apparent that, if a distance or écart is to be defined satisfying certain intuitive ideas of distance and also satisfying the condition that ordinary convergence is to imply convergence in terms of the distance, then the general functions must be so classified that all of the functions $f(x) \equiv a$ will be regarded as identical.

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* Fundamenta Mathematicae, vol. 9, p. 34.

A NEIGHBORHOOD TREATMENT OF GENERAL TOPOLOGICAL SPACES*

BY W. O. GORDON

In this paper we deal with all the subsets of a space R of elements called points. Each point p of R may have associated with it certain subsets of R called *neighborhoods of* p so that it is determined by some rule whether or not any particular set is a neighborhood of any particular point. In the most general case no assumptions are made such as that every point has at least one neighborhood associated with it, or that the point p is an element of the neighborhood associated with it.

The purpose of this paper will be to consider (1) the various ways of defining limit point in terms of neighborhoods; (2) what properties must be assumed concerning the neighborhoods in order that limit point have certain well known properties. We shall start with the following new definition of limit point which, although somewhat peculiar in character, is found to be most convenient for the case of the general topological space.†

DEFINITION A. A point p is said to be a *limit point* of a set E if every neighborhood of p that contains $C(E)$‡ contains at least one point of E.

DEFINITION B. A point p is said to be *interior* to the set E if it is not a limit point of $C(E)$.

THEOREM 1. *The set N is a neighborhood of the point p if and only if p is interior to N.*

PROOF. (1) Suppose p were not interior to N; then p would be a limit point of $C(N)$ and hence every neighborhood of p that contains the complement of $C(N)$ must contain a point of $C(N)$. Now N is a neighborhood of p by hypothesis, but it contains no point of $C(N)$, which gives us a contradiction. (2) If p is not a limit point of $C(N)$, then there is a neighborhood of p which contains N but no point of $C(N)$, hence N is the neighborhood.

* Presented to the Society, June 22, 1933.

† M. Fréchet, *Les Espaces Abstraites*, p. 166; E. W. Chittenden, Transactions of this Society, vol. 31 (1929), pp. 290–321.

‡ The symbol $C(E)$ means the complement of the set E. The whole definition implies that p is a limit point of E if $C(E)$ is not a neighborhood of p.

From this theorem it can be seen that with Definition A the general topological space can be considered as a neighborhood space. For if we take for the neighborhoods of each point p all sets to which p is interior, then by Theorem 1, Definition A will define limit point correctly.

DEFINITION C. A space is said to be *monotonic** if it has the property that for every point p and set A, if p is a limit point of A it is a limit point of every set containing A. This can be written in symbols:†

If $A<B$, then $A'<B'$, or $(A+B)'>A'+B'$.

THEOREM 2. *The space R is monotonic if and only if it has the property that if the set N is a neighborhood of the point p, so is every set containing N.*

The proof of Theorem 2 follows easily from Definition C and will be omitted.

DEFINITION D. A space is said to have the *partition property*‡ if whenever p is a limit point of $(A+B)$, it is a limit point of either A or B. In symbols $(A+B)'<A'+B'$.

THEOREM 3. *A necessary and sufficient condition that a space R have the partition property is that if M and N are neighborhoods of the point p, so is their logical product $M\cdot N$.*

PROOF. First, suppose p were not a limit point of either A or B. Then $C(A)$ and $C(B)$ would be neighborhoods of p and so would their logical product $C(A)\cdot C(B)=C(A+B)$; then $C(A+B)$ would be a neighborhood of p and consequently p could not be a limit point of $A+B$, contrary to hypothesis. Secondly, suppose $M\cdot N$ were not a neighborhood; then p is a limit point of $C(M\cdot N)$ which is equal to $C(M)+C(N)$, and hence p is a limit point either of $C(M)$ or $C(N)$, so that neither M nor N could be a neighborhood of p, contrary to hypothesis.

DEFINITION E. A set is *closed* if it contains all of its limit points.

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* Dorothy McCoy, Tôhoku Mathematical Journal, vol. 33 (1930), pp. 88–116.

† The symbol $<$ in $A<B$ means "A is contained in B". The symbol A' means the set of limit points of the set A. The symbol $>$ in $(A+B)'>A'+B'$ means that $(A+B)'$ includes $A'+B'$.

‡ See Dorothy McCoy, loc. cit.

DEFINITION F. A set is said to be *open* if every point of it is interior to it. Thus an open set is the complement of a closed set, and conversely.

DEFINITION G. The set of all limit points of the set A is called the *derived set*, A'.

DEFINITION H. If all the derived sets in the space R are closed, that is, if $(A')' < A'$, the space R is said to be *accessible*.

THEOREM 4. *A space R is accessible if and only if it is true that whenever N is a neighborhood of a point p, then the set of all points of which N is a neighborhood is also a neighborhood of the point p.*

PROOF. First, let I be the set of all points of which N is a neighborhood. By Theorem 1, I consists of all points that are not limit points of $C(N)$; hence $C(I) = (C(N))'$ which is closed by hypothesis. Hence $C(I)$ is closed and therefore I is open, and, consequently, I is a neighborhood of p. Secondly, suppose p is a limit point of A' but not of A; then $C(A)$ is a neighborhood of p, and so I, which is the set of all points of which $C(A)$ is a neighborhood, is a neighborhood of p by hypothesis, but $I = C(A')$ and hence p is not a limit point of A', contrary to hypothesis.

COROLLARY. *The space R is accessible if and only if the set of all points interior to any set is an open set.*

DEFINITION I. A space R is said to have the *non-singular** property if whenever p is a limit point of the set A it is a limit point of $A - p$.

THEOREM 5. *The space R is non-singular if and only if whenever p is a limit point of the set A, every neighborhood of p which contains $C(A)$ contains a point of A other than p.*

PROOF. First, suppose there exists a neighborhood of p which includes $C(A)$ but contains no point x of A other than p. The complement of this neighborhood is equal to A or $A - p$; p is a limit point of the complement of this neighborhood N and therefore N could not be a neighborhood of p, contrary to assumption. Secondly, suppose the space R were non-singular; then $C(A - p)$ which is equal to $C(A) + p$ would be a neighborhood of p and therefore contain a point of A other than p, but this is impossible.

* Dorothy McCoy, loc. cit.

THEOREM 6. *The space R is monotonic if and only if whenever p is a limit point of the set A, every neighborhood of the point p contains a point of A.*

This follows easily and the proof will be omitted.

DEFINITION J. A space R is said to have the *infinite* property if whenever p is a limit point of the set A it is a limit point of the set $A-x$, x being a set containing a single point.

THEOREM 7. *The space R has the infinite property if and only if whenever p is a limit point of the set A, every neighborhood of p which contains the complement of A contains an infinite number of points of A.*

PROOF. First, suppose there exists a neighborhood of p, N_p, which contains $C(A)$ and has only a finite number of points of the set A. Call this finite set F. Now p is a limit point of $A-F$ by hypothesis and $N_p=C(A-F)$ which is equal to $C(A)+F$, and therefore N_p must contain a point of $A-F$ by the definition of limit point, but this is impossible since N_p contains only $C(A)$ and the finite set F. Secondly, suppose p is a limit point of $A+F$ (the set F being finite) but not of A. Then $C(A)$ is a neighborhood of p and contains $C(A+F)$ and hence by hypothesis must have an infinite set of points of $A+F$, but these must be in F, which is finite, and so we have a contradiction.

DEFINITION K. Two systems of neighborhoods $\{M_p\}$ and $\{N_p\}$ of a point p are said to be *equivalent* if every M_p contains an N_p and every N_p contains an M_p.

THEOREM 8. *If in a monotonic space R, a system of neighborhoods $\{M_p\}$ of the point p is equivalent to the set of all neighborhoods $\{N_p\}$ of p, then p is a limit point of the set S if and only if every M_p contains a point of the set S.*

This follows easily from Theorem 6 and the proof will be omitted.

CONCLUSION. It is possible to define limit point in terms of neighborhood in the following four ways, each of which is, in a sense, more special than the preceding.

1. A point p is a limit point of the set S if every neighborhood of p which contains the complement of S also contains a point of the set S.

2. A point p is a limit point of the set S if every neighborhood of p contains a point of the set S.

3. A point p is a limit point of the set S if every neighborhood of p contains a point of the set S other than p.

4. A point p is a limit point of the set S if every neighborhood of p contains an infinite number of points of the set S.

Definition 1 is the one we have used throughout. Although unusual in its form, it has the advantage that it implies no properties of limit point and hence can be used in the most general topological spaces. In particular it does not imply the monotonic property, which distinguishes it from all previous neighborhood definitions of limit point. Hence it can be used in connection with the notion of sequential limit point, which does not have the monotonic property, or with the notion of boundary point. Other non-monotonic relations between point and set will suggest themselves, which can now be treated in terms of neighborhood, using Definition 1.

Definition 2 is the simplest of these definitions. It obviously implies the monotonic property. Theorem 6 shows that every monotonic space can be treated as a neighborhood space using this definition of limit point. Although Definition 2 is very simple and leads to a very neat treatment of the most general monotonic space, it does not seem to have been considered in the literature. This definition acquires added importance when it is remembered that Chittenden (loc. cit.) has shown that in the most general topological space one can redefine limit point so as to make the space monotonic.

Definition 3 is one of the usual ways of defining limit point in terms of neighborhood and is used by Fréchet* for his very general V-spaces. If we use this definition the space is necessarily monotonic and non-singular. In connection with Theorem 1, Theorems 5 and 6 show that in any monotonic and non-singular space we can define limit point in terms of neighborhood using Definition 3.

Similarly Definition 4 implies the monotonic and infinite properties, and Theorems 6 and 7 show that this definition, with neighborhoods defined as suggested by Theorem 1, can be used in the most general space having these two properties.

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* *Les Espaces Abstraites*, p. 172.

NOTE ON CUBIC SURFACES IN THE GALOIS FIELDS OF ORDER 2^n*

BY A. D. CAMPBELL

Let us consider the cubic surface with the equation

$$\begin{aligned} f(x, y, z, w) \equiv\ & a_0w^3 + (b_0x + b_1y + b_2z)w^2 + (c_0x^2 + c_1y^2 \\ & + c_2z^2 + c_3yz + c_4zx + c_5xy)w + (d_0x^3 \\ & + d_1y^3 + d_2z^3 + d_3y^2z + d_4yz^2 + d_5x^2y \\ & + d_6xy^2 + d_7x^2z + d_8xz^2 + d_9xyz) = 0, \end{aligned} \tag{1}$$

whose coefficients and variables represent numbers in a Galois field of order 2^n. The first polar (or polar quadric) of any point $P'(x', y', z', w')$ with respect to (1) is

$$\begin{aligned} & (d_0x' + d_5y' + d_7z' + c_0w')x^2 + (d_6x' + d_1y' + d_3z' \\ & + c_1w')y^2 + (d_8x' + d_4y' + d_2z' + c_2w')z^2 + (b_0x' + b_1y' \\ & + b_2z' + a_0w')w^2 + (d_9z' + c_5w')xy + (d_9y' + c_4w')xz \\ & + (c_5y' + c_4z')xw + (d_9x' + c_3w')yz + (c_5x' + c_3z')yw \\ & + (c_4x' + c_3y')zw = 0. \end{aligned} \tag{2}$$

The second polar of P' with respect to (1) can be obtained from (2) by interchanging x' and x, y' and y, z' and z, w' and w. The polar quadric of (0, 0, 0, 1) is

$$c_0x^2 + c_1y^2 + c_2z^2 + a_0w^2 + c_3yz + c_4zx + c_5xy = 0. \tag{3}$$

The second polar of (0, 0, 0, 1) is

$$b_0x + b_1y + b_2z + a_0w = 0. \tag{4}$$

The Hessian of (1) is

$$\begin{aligned} & (d_9z + c_5w)(c_4x + c_3y) + (d_9y + c_4w)(c_5x + c_3z) \\ & + (c_5y + c_4z)(d_9x + c_3w) \equiv 0. \end{aligned} \tag{5}$$

We note in fact, that the first polar of (0, 0, 0, 1) with respect to (3) vanishes identically; whereas the first polars of (1, 0, 0, 0), (0, 1, 0, 0), and (0, 0, 1, 0), respectively, are

* Presented to the Society, October 29, 1932.

$$c_5y + c_4z = 0, \quad c_5x + c_3z = 0, \quad c_4x + c_3y = 0,$$

with

$$\begin{vmatrix} 0 & c_5 & c_4 \\ c_5 & 0 & c_3 \\ c_4 & c_3 & 0 \end{vmatrix} = 0,$$

all of which planes intersect in a line whose points have the coordinates (c_3, c_4, c_5, k) where k is arbitrary.

The statements in the preceding sentences are true only for the Galois fields of order 2^n. We shall now study the straight lines on the cubic surface (1) by a method that is valid also for cubic surfaces in any Galois field or in the ordinary real or complex three-dimensional space. The argument used is briefer than such a discussion as that in Salmon-Rogers, *Analytic Geometry of Three Dimensions*. To determine whether or not a line

$$z = \lambda_3x + \mu_3y, \quad w = \lambda_4x + \mu_4y \tag{6}$$

lies on the surface (1), we substitute in (1) for x, y, z, w respectively the variables

$$y_1 = x + 0y, \quad y_2 = 0x + y, \quad y_3 = \lambda_3x + \mu_3y, \quad y_4 = \lambda_4x + \mu_4y. \tag{7}$$

We get the Taylor expansion

$$\begin{aligned} f(y_1, y_2, y_3, y_4) &= f(x + 0y, 0x + y, \lambda_3x + \mu_3y, \lambda_4x + \mu_4y) \\ &= x^3f(1, 0, \lambda_3, \lambda_4) + x^2y\left(0\frac{\partial'}{\partial y_1} + 1\frac{\partial'}{\partial y_2} + \mu_3\frac{\partial'}{\partial y_3} + \mu_4\frac{\partial'}{\partial y_4}\right)f \\ &\quad + \frac{xy^2}{2}\left(0\frac{\partial'}{\partial y_1} + 1\frac{\partial'}{\partial y_2} + \mu_3\frac{\partial'}{\partial y_3} + \mu_4\frac{\partial'}{\partial y_4}\right)^{(2)}f \\ &\quad + y^3f(0, 1, \mu_3, \mu_4) = 0. \end{aligned} \tag{8}$$

By $\partial'/\partial y_i$ we mean a derivative with respect to y_i wherein the variables y_1, y_2, y_3, y_4 are subsequently replaced respectively by x, $0x$, λ_3x and λ_4x. By the expression $(0\partial'/\partial y_1 + \cdots + \mu_4\partial'/\partial y_4)^{(k)}f$ we mean the usual collection of terms containing kth derivatives of f with the factors in x and y removed. The number 2 in the denominator of $xy^2/2$ cancels the 2 in the coefficient.

If the line (6) lies entirely on (1) the coefficients of the terms in x^3, x^2y, xy^2, and y^3 in (8) must vanish. Geometrically this

means that the points $P_1(0, 1, \mu_3, \mu_4)$ and $P_2(1, 0, \lambda_3, \lambda_4)$ in which the line (6) cuts the planes $x=0$ and $y=0$ must both lie on (1), also P_1 must lie on the first and second polar of P_2 with respect to (1).

If we wish to find all such lines on (1) we must find the equation of condition ($K=0$) that a first degree equation in y, z, w (which is the line in which the second polar of $P'(x', 0, z', w')$ with respect to (1) cuts $x=0$) and a second degree equation in y, z, w (which is the conic in which the first polar of P' cuts $x=0$) and a third degree equation in y, z, w (which is the cubic curve in which (1) cuts $x=0$) shall have a common solution $P''(0, y'', z'', w'')$. Then we must solve $K=0$ (an equation in x', z', w') simultaneously with the equation obtained from (1) by putting therein $x=x'$, $y=0$, $z=z'$, $w=w'$. The result is too complicated to quote here. There turn out to be twenty-seven such lines.

To discuss the double points on (1) we use a method suitable only for the Galois fields of order 2^n. By suitable transformations we can reduce (1) to a form with $c_3=c_4=c_5=0$. We shall suppose this has been done. To find the condition for double points on (1) and the number of such double points, we take the first polars of (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), and (0, 0, 0, 1) and find the condition for these polars to have common solutions and find also how many such solutions they can have. The desired equations are

$$
\begin{aligned}
b_0w^2 + d_0x^2 + d_6y^2 + d_8z^2 + d_9yz &= 0,\\
b_1w^2 + d_5x^2 + d_1y^2 + d_4z^2 + d_9xz &= 0,\\
b_2w^2 + d_7x^2 + d_3y^2 + d_2z^2 + d_9xy &= 0,\\
a_0w^2 + c_0x^2 + c_1y^2 + c_2z^2 &= 0.
\end{aligned}
\tag{9}
$$

If $d_9 \neq 0$, we eliminate w^2 from the first three equations, using the fourth equation to do so. Then we square the first and third of the three resulting equations and use the second equation to eliminate y^2 from these two squared equations. We now have two equations in x and z of the form

$$
ax^4 + bx^2z^2 + cxz^3 + dz^4 = 0,\ \lambda x^4 + \mu x^3z + \nu x^2z^2 + \rho z^4 = 0. \tag{10}
$$

From the form of the equations (10) we see that they can have only three, two, one, or no common roots in x/z. Hence in gen-

eral (1) can have (if at least one of the terms in xyz, xzw, xyw, yzw is not missing) three, two, one, or no double points. All such double points will lie on the plane

$$a_0^{1/2}w + c_0^{1/2}x + c_1^{1/2}y + c_2^{1/2}z = 0. \tag{11}$$

We note as an exception to the above discussion of double points on (1) that if $a_0=c_0=c_1=c_2=c_3=c_4=c_5=0$, then we can use the first equation of (9) to eliminate w^2 from the next two equations and the two resulting quadratic equations in x, y, z may have four common solutions. Hence (1) may in this case have four double points.

If $d_9=0$, the equations (9) have just one common solution if and only if

$$\begin{vmatrix} d_0 & d_6 & d_8 & b_0 \\ d_5 & d_1 & d_4 & b_1 \\ d_7 & d_3 & d_2 & b_2 \\ c_0 & c_1 & c_2 & a_0 \end{vmatrix} = 0. \tag{12}$$

Therefore if $c_3=c_4=c_5=d_9=0$ in (1) and if (12) is satisfied, then the cubic surface (1) has just one double point, otherwise it has no double points. We note that if and only if $c_3=c_4=c_5=d_9=0$, every first polar (2) is a double plane.

If $a_0=c_0=c_1=c_2=c_3=c_4=c_5=d_9=0$, we see from the preceding paragraph that (1) has a line of double points. This holds also if for any other reason all the third-order minors of (12) vanish.

To find the equation of (1) in plane coordinates we assume that $c_3=c_4=c_5=0$ and proceed as follows. Solving $\delta w=\alpha x+\beta y+\gamma z$ simultaneously with (1), we obtain the equation

$$\lambda_0x^3 + \lambda_1y^3 + \lambda_2z^3 + \lambda_3x^2y + \lambda_4xy^2 + \lambda_5x^2z + \lambda_6xz^2 + \lambda_7y^2z + \lambda_8yz^2 + d_9xyz = 0. \tag{13}$$

We note the presence of d_9 in (13). We note that

$$\lambda_0 = a_0\alpha'^3 + b_0\alpha'^2 + c_0\alpha' + d_0, \quad \lambda_3 = a_0\alpha'^2\beta' + b_1\alpha'^2 + c_0\beta' + d_5,$$

where $\alpha'=\alpha/\delta$, $\beta'=\beta/\delta$, $\gamma'=\gamma/\delta$, and the other λ's have similar values in α', β', γ'. We seek now the equation of condition for the first polars of (1, 0, 0), (0, 1, 0), and (0, 0, 1) with respect to (13) to have a common solution. This equation of condition

will be the equation of (1) in the plane coordinates α, β, γ, δ. The desired first polars are

$$\begin{aligned} &\lambda_0x^2+\lambda_1y^2+\lambda_6z^2=d_9yz, \quad \lambda_3x^2+\lambda_1y^2+\lambda_8z^2=d_9xz, \\ &\lambda_5x^2+\lambda_7y^2+\lambda_2z^2=d_9xy. \end{aligned} \tag{14}$$

If $d_9\neq 0$, we multiply together the first two equations of (14) and in the resulting equation replace d_9xy in the term $d_9^2xyz^2$ by $\lambda_5x^2+\lambda_7y^2+\lambda_2z^2$. Then we multiply together the first and third equations of (14) and in the resulting equation replace d_9xz in the term $d_9^2xy^2z$ by $\lambda_3x^2+\lambda_1y^2+\lambda_8z^2$. Finally we multiply together the last two equations of (14) and in the resulting equation replace d_9yz in the term $d_9^2x^2yz$ by $\lambda_0x^2+\lambda_4y^2+\lambda_6z^2$.

Then we take the square root of each of these three equations we have obtained and we get

$$\begin{aligned} &(\lambda_0\lambda_3)^{1/2}x^2+(\lambda_1\lambda_4)^{1/2}y^2+(\lambda_5\lambda_6+d_9\lambda_2)^{1/2}z^2 \\ &\quad +\Lambda_2{}^{1/2}xy+(\Lambda_7+d_9\lambda_5)^{1/2}xz+(\Lambda_5+d_9\lambda_7)^{1/2}yz=0, \\ &(\lambda_0\lambda_5)^{1/2}x^2+(\lambda_4\lambda_7+d_9\lambda_1)^{1/2}y^2+(\lambda_6\lambda_2)^{1/2}z^2 \\ &\quad +(\Lambda_8+d_9\lambda_3)^{1/2}xy+\Lambda_1{}^{1/2}xz+(\Lambda_3+d_9\lambda_8)^{1/2}yz=0, \\ &(\lambda_3\lambda_5+d_9\lambda_0)^{1/2}x^2+(\lambda_1\lambda_7)^{1/2}y^2+(\lambda_8\lambda_2)^{1/2}z^2 \\ &\quad +(\Lambda_6+d_9\lambda_4)^{1/2}xy+(\Lambda_4+d_9\lambda_6)^{1/2}xz+\Lambda_0{}^{1/2}yz=0, \end{aligned} \tag{15}$$

wherein Λ_i is the minor of λ_i in Δ_0, the determinant of the coefficients of x^2, y^2, z^2 in the three equations of (14).

If we look upon the six equations of (14) and (15) as linear equations in x_1, x_2, x_3, x_4, x_5, x_6, respectively, equal to x^2, y^2, z^2, yz, xz, xy, the equation of condition for them to have a common solution in these six variables is $\Delta=0$, where Δ is the determinant of the coefficients of the six variables. The equation $\Delta=0$ is at most of the twelfth degree in α, β, γ, δ.

If $d_9=0$, we see from (14) that the equation of (1) in plane coordinates is $\Delta_0=0$, which is at most of the ninth degree in α, β, γ, δ. We note that Δ reduces to Δ_0 for $d_9=0$.

From the above discussion we see that (1) has ordinarily an equation of the twelfth degree in plane coordinates, but that if all the terms in xyz, yzw, xyw, xzw are missing from (1), then the equation of (1) in plane coordinates is only of the ninth degree at most.

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ON A METHOD OF COMPARISON FOR STRAIGHT-LINE NETS*

BY LOUISE D. CUMMINGS

1. *Introduction.* The problem which led to this investigation of the structure of systems of straight lines in a plane was undertaken for the purpose of reclassifying, if possible, the 36 Weber-Aronhold sets of seven real double-tangents of a plane quartic curve. Different ways of constructing straight-line nets are known, but easy and conclusive methods for comparing systems apparently different are at least not well known.

We exhibit here a new method of comparison for straight-line nets and apply it to show all non-equivalent types for $n=6$, 7, and all *heptagonal* types for $n=8$.

2. *Non-Equivalent Sets of n Lines.* A set of real lines, finite in number, in the plane of projective geometry, divides the plane into convex polygons. If no three of the lines meet in a point, then by Euler's equation n lines form $(n^2-n+2)/2$ polygons. Two sets of n lines are *equivalent* when a one-to-one relation exists between the lines and polygons of the two sets. If $n<6$, the sets are equivalent, but for $n\geqq 6$, four or more non-equivalent sets are known to exist.†

3. *Method of Construction.* Since all sets of five lines are equivalent, any five lines in a set may be selected as the initial five. Adding a sixth line, we construct systems of six lines whose equivalence or non-equivalence is tested by this new method of comparison, and only four different types exist. Employing these four types of six lines as initial sets and adding a seventh line we find that only eleven non-equivalent sets of seven lines exist. Among the eleven non-equivalent sets of seven lines, one set contains a convex heptagon. From this set by adding an eighth line we derive fifteen non-equivalent sets of eight lines, designated as *heptagonal* sets of eight lines.

* Presented to the International Congress of Mathematicians, Zurich, September 5, 1932.

† H. S. White, this Bulletin, vol. 38 (1932), p. 59; and L. D. Cummings, this Bulletin, vol. 38 (1932), p. 105; also vol. 38 (1932), p. 700.

4. *Notation and Method of Comparison.* Let l_1 and l_2, copunctual at p, be any two of a net of n lines. On these two lines lie $(2n-3)$ of the $n(n-1)/2$ points of intersection of the net. The two lines divide the plane into two regions I and II *inside* of which the remaining $(n-2)(n-3)/2$ points of intersection lie.

The *index of the point p* shows the exact distribution in regions I and II of these residual points of intersection. The $n(n-1)/2$ indices of all the points of intersection of the n lines are determined, like indices are collected together, and the tabulated result is designated as the *index of the system.* The index of any system is a unique characterization of that system and furnishes probably a sufficient test for the equivalence or non-equivalence of that system with any other system.

5. *Systems of Six Lines.* For brevity this new method of comparison is shown in its application to systems of six lines. Five

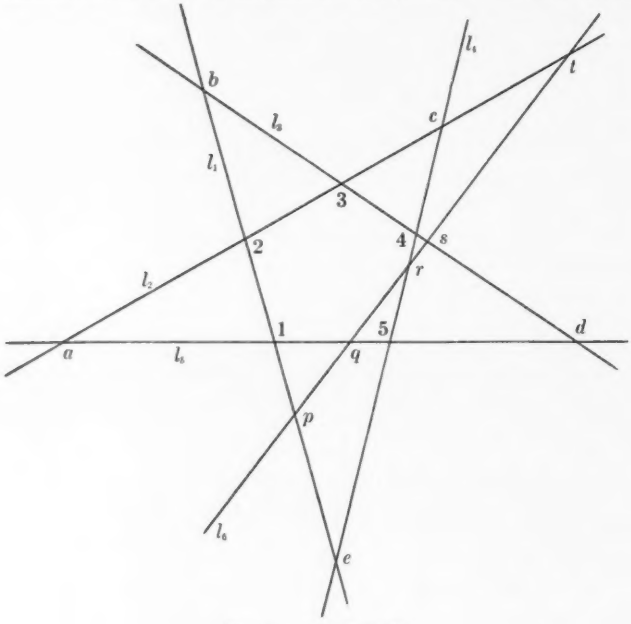

FIG. 1. SYSTEM (1).

lines divide the plane into eleven convex polygons, one pentagon, five triangles, and five quadrilaterals. A pentagon with sides produced indefinitely and all ten intersections in the finite plane is selected as typical arrangement for $n=5$. Let there be given a sixth line or secant crossing (i) two consecutive sides of the pentagon as in Figure 1, System (1).

Designate the vertices of the pentagon in order 1, 2, 3, 4, 5; the vertices of the five triangles in the same cyclic order a, b, c, d, e; and the intersections of the sixth line l_6 with the sides of the pentagon p, q, r, s, t. Let the sides 12, 23, 34, 45, 51 of the pentagon be l_1, l_2, l_3, l_4, l_5.

For $n=6$, we have $n(n-1)/2=15$, $(2n-3)=9$, and there are $(n-2)(n-3)/2=6$ residual points. At point 1 the two lines l_1 and l_5 divide the plane into two regions I and II. Region I contains none of the residual points of intersection and region II contains six of them, and point 1 has the *index* (0, 6), written 1 (0, 6); the point d has *index* (2, 4) or $d(2, 4)$; $p(3, 3)$; and this system has no points with index (1, 5). The index (i, j) of any point is always written with the smaller number first and may be abbreviated to i as in the column-headings of Table III.

Figure 1, System (1) contains six points 1, 2, 3, 4, r, q with index (0, 6); no points with index (1, 5); three points d, e, t with index (2, 4) and six points a, b, c, s, 5, p with index (3, 3). These results are given in tabular form as

System	Indices			
	(0, 6)	(1, 5)	(2, 4)	(3, 3)
(1)	6	0	3	6

The group of order 12 generated by

$$S_1 \equiv (d\,e\,t)(1\,2\,3\,4\,r\,q)(a\,b\,c\,s\,5\,p),$$

$$S_2 \equiv (1\,3)(a\,b)(c\,p)(d)(e\,t)(4\,q)(5\,s)(2)(r)$$

transforms the system (1) into itself. This method of indices simplifies the determination of all substitutions which transform a system into itself or into any equivalent system.

The investigation of all possible positions for the sixth line or secant shows only three additional non-equivalent systems. In the following Table I the indices for each of the four non-equivalent systems are given and also, incidentally, the actual polyg-

TABLE I
THE PLANE FIGURE OF SIX REAL LINES

SYSTEM	INDICES				POLYGONS			
	(0, 6)	(1, 5)	(2, 4)	(3, 3)	S_6	S_5	S_4	S_3
(1)	6	0	3	6	1	0	9	6
(2)	4	4	4	3	0	2	8	6
(3)	3	6	6	0	0	3	6	7
(4)	0	15	0	0	0	6	0	10

onal division of the projective plane, satisfying of course the Euler equation

$$s_6 + s_5 + s_4 + s_3 = \frac{n^2 - n + 2}{2},$$

where s_i, the number of polygons of i sides, is tabulated for each of the four systems.

6. *Systems of Seven Lines.* To form the systems of seven lines, each of the four non-equivalent systems of six lines of

TABLE II
THE PLANE FIGURE OF SEVEN REAL LINES

SYSTEM	INDICES						POLYGONS				
	(0,10)	(1, 9)	(2, 8)	(3, 7)	(4, 6)	(5, 5)	S_7	S_6	S_5	S_4	S_3
(1)	7	0	0	0	14	0	1	0	0	14	7
(7)	1	6	6	4	2	2	0	1	5	5	11
(8)	3	0	12	0	6	0	0	1	3	9	9
(3)	4	2	3	6	4	2	0	1	2	11	8
(2)	5	2	0	6	4	4	0	1	1	13	7
(11)	0	9	3	3	6	0	0	0	6	6	10
(5)	2	6	3	2	6	2	0	0	5	8	9
(6)	3	4	4	2	2	6	0	0	4	10	8
(10)	2	4	5	6	2	2	0	0	4	10	8
(4)	4	2	3	4	8	0	0	0	3	12	7
(9)	3	3	6	0	6	3	0	0	3	12	7

When, in a set of seven lines, any six form a convex hexagon, the system is designated as *hexagonal.*

The eight systems (1), (2), (3), (4), (5), (6), (7), (8) are hexagonal and the three systems (9), (10), (11) are non-hexagonal.

Table I is used as a basic system and all possible positions for the seventh line or secant are investigated.

Table II exhibits the indices for each of the eleven non-equivalent systems of seven real lines in the plane, and shows also the polygonal division of the plane for each of these systems. If in a system of seven lines any six form a convex hexagon, the system is designated as an *hexagonal* system of seven lines. In Table II the eight systems (1), (2), (3), (4), (5), (6), (7), (8) are hexagonal systems derived from the basic system (1) of Table I. When systems (2), (3), (4) of Table I are used as basic systems only three additional non-equivalent systems (9), (10), (11) of Table II are obtained and these are designated the *pentagonal* systems of seven lines.

Systems (6) and (10) in Table II have different indices but the same invariant numbers in the polygonal divisions, and show that equality of polygonal numbers is a necessary but not a sufficient condition for the equivalence of two systems. A comparison of systems (4) and (9) shows the same fact.

7. *Systems of Eight Lines.* To determine the non-equivalent systems of eight lines I assume the eleven non-equivalent systems of Table II as bases and examine all possible positions of the eighth line. A system of eight lines is designated as *heptagonal* if any seven of the lines form a convex heptagon. Table III shows the indices and the polygonal numbers for each of the fifteen non-equivalent *heptagonal* systems of eight lines derived from the base system (1) of Table II. In Table III the three non-equivalent systems (15), (12), and (14) have the same invariant polygonal numbers but quite different indices.

8. *Conclusion.* We have now two distinct methods of comparison: that of *marks* developed by H. S. White* and this of *indices*, both easy of application, which have given reliable and accordant results, positive as well as negative, in all cases in which they have been employed. Both methods lead to the easy discovery of the substitutions which transform any system into itself or into an equivalent system. The two methods have been applied to 36 Weber-Aronhold sevens and show a new distribution of the same into 11 non-equivalent subdivisions.

* *The plane figure of seven real lines*, this Bulletin, vol. 38 (1932), pp. 59–65.

TABLE III

HEPTAGONAL SYSTEMS OF EIGHT LINES IN A PLANE

SYSTEM	INDEX								POLYGONS					
	0	1	2	3	4	5	6	7	S_8	S_7	S_6	S_5	S_4	S_3
(1)	8	0	0	0	0	8	4	8	1	0	0	0	20	8
(11)	0	7	0	7	7	0	0	7	0	1	0	7	7	14
(15)	2	6	0	3	3	5	7	2	0	1	0	5	11	12
(12)	2	3	4	2	8	2	2	5	0	1	0	5	11	12
(14)	2	4	4	0	6	4	6	2	0	1	0	5	11	12
(13)	4	0	4	3	5	6	0	6	0	1	0	3	15	10
(3)	5	2	1	1	3	5	8	3	0	1	0	2	17	9
(4)	5	1	2	2	2	6	8	2	0	1	0	2	17	9
(2)	6	2	0	0	2	6	7	5	0	1	0	1	19	8
(6)	2	5	3	0	5	2	6	5	0	0	1	5	12	11
(7)	3	4	1	3	3	3	7	4	0	0	1	4	14	10
(5)	3	4	2	1	4	3	6	5	0	0	1	4	14	10
(9)	4	1	4	2	2	6	2	7	0	0	1	3	16	9
(10)	4	4	0	1	4	3	6	6	0	0	1	3	16	9
(8)	5	1	1	3	3	5	4	6	0	0	1	2	18	8

A system of 8 lines is *heptagonal* if any seven of the lines form a convex heptagon.

The intrinsic difference between Weber's basis of classification of sets of seven lines and that investigated in this paper is that Weber relates the heptads to a quartic curve and this paper studies them as sets of lines, without relation to any other figure.

This method of analysis of straight-line nets by the indices, as herein extended to all the points of intersection of the system, regardless of their relation to the vertices of a pentagon, hexagon, or other basal polygon, is applicable to any number n of straight lines and moreover is not restricted to the case that only two straight lines pass through a point.

VASSAR COLLEGE

REDUCIBLE DIOPHANTINE SYSTEMS

BY E. T. BELL

1. *Reducibility.* The algorithm of reciprocal arrays, developed in a previous paper,* can be applied to obtain the complete solutions in integers of a larger class of diophantine equations. In this note the application to equations reducible to Type VI of the previous paper is described; the most general Type VII can be treated similarly. Details, particularly in the examples in §3, which follow immediately from the theory of reciprocal arrays, which was fully discussed, are omitted. By *solution* is meant *all sets of rational integers satisfying a given diophantine system.*

A point which was inadvertently omitted from the paper on arrays must first be supplied. Let, for the moment,

$$X_1, \cdots, X_n; Y_1, \cdots, Y_m; \cdots; W_1, \cdots, W_s \tag{1}$$

be $n+m+\cdots+s$ independent variables, finite in number; let $a, b, \cdots, c$ be arbitrary constant integers $\neq 0$, and let the $a_i, b_j, \cdots, c_k$ be arbitrary constant integers >0. To find the solution of

$$aX_1^{a_1}\cdots X_n^{a_n} = bY_1^{b_1}\cdots Y_m^{b_m} = \cdots = cW_1^{c_1}\cdots W_s^{c_s}, \tag{2}$$

when $|ab\cdots c|$ *is greater than unity* (omitted from the previous paper), we treat $a, b, \cdots, c$ first as independent variables, and solve for them along with the X's, Y's, $\cdots$, W's. In those solutions in which no $X, Y, \cdots, W$ takes the value zero, $a, b, \cdots, c$ will thus be given as products of powers of the parameters.† Thus each of the constant integers $a, b, \cdots, c$ is exhibited as a power product of a certain type, say

$$a = \alpha_1^{p_1}\cdots\alpha_r^{p_r},\ b = \beta_1^{q_1}\cdots\beta_t^{q_t}, \cdots, c = \gamma_1^{h_1}\cdots\gamma_u^{h_u},$$

where the Greek letters are certain of the parameters. Hence

* E. T. Bell, *Reciprocal arrays and diophantine analysis*, American Journal of Mathematics, vol. 55, January, 1933, pp. 50–66; M. Ward, *A type of multiplicative diophantine system*, ibid., pp. 67–77.

† A *parameter* here is a variable ranging over integer values, zero excluded.

these parameters in the solution are constrained to take only a finite number of distinct sets of constant values, since $a, b, \cdots, c$ are constants.* Each such set defines a class of solutions. If $N(n; d_1, \cdots, d_g)$ is the total number of decompositions of n in the form $n = \xi_1^{d_1} \cdots \xi_g^{d_g}$, the number of classes in the solution of (2) is

$$N(a; p_1, \cdots, p_r)N(b; q_1, \cdots, q_t) \cdots N(c; h_1, \cdots, h_u).$$

We shall now define reducibility as applied to diophantine systems. Let

$$(3) \qquad x_1, \cdots, x_n; y_1, \cdots, y_m; \cdots; w_1, \cdots, w_s$$

be $n+m+\cdots+s$ independent variables. If a diophantine system in the independent variables (3) is equivalent to the system (2), where now the variables (1) are linear homogeneous functions of the variables (3) with integer coefficients (there is no gain in generality in rational coefficients), of the following kind,

(4) the X's are functions of the x's alone, the Y's of the y's alone, $\cdots$, the W's of the w's alone,

(5) the respective determinants of the X's, Y's, $\cdots$, W's are $A, B, \cdots, C$, and $AB \cdots C \neq 0$,

we shall say that the diophantine system is *reducible*.

THEOREM. *The solution of a reducible diophantine system is given by an application of reciprocal arrays followed, if* $|AB \cdots C| > 1$, *by a simple enumeration of cases according to congruences moduli* $A, B, \cdots, C$, *and the solution by classical methods of certain linear homogeneous diophantine equations with given constant coefficients.*

Consider first the sets of integers contributed to the solution when none of the $X, Y, \cdots, W$ is zero. Then, since $AB \cdots C \neq 0$, the solution of the system (2), in which the variables are as in (1), enables us to solve algebraically for the variables (3). This expresses the variables (3) of the original system as sums of

* It can be shown that resolutions of $a, b, \cdots, c$ in the above type exist. For example, if $a = 11$, the resolution $a = \alpha_1^2 \alpha_2^3$ can not occur, but $a = \alpha_1^2 \alpha_2^3 \alpha_3$ may, and $|\alpha_1| = |\alpha_2| = 1$, $\alpha_3 = 11$ are the only possibilities.

power-products of parameters with rational coefficients; the coefficients of the x's, y's, $\cdots$, w's have the respective common denominators $A, B, \cdots, C$. That an infinity of sets of integers $x_i, y_j, \cdots, w_k$ satisfying the original system exist is evident, since each parameter may be replaced by $AB \cdots C$ times itself. The finding of all such sets, and hence the first stage of the solution, becomes a straight-forward exercise in linear congruences, moduli $A, B, \cdots, C$, on replacing the several powers of parameters by their possible forms to the given moduli. This further separates the sets in the solution into classes. Since $A, B, \cdots, C$ are constants, all the classes can be found in a finite number of steps, explicitly.

The sets contributed when some of the $X, Y, \cdots, W$ are zero are found by solving sets of linear homogeneous equations. Thus, for example, singling out the X's, if $n=1$, then $x_1=0$, and at least one of each of the Y's, $\cdots$, W's must vanish, while the rest may take any arbitrary integer values. Suppose $n>1$. Then, since $A \neq 0$, not all the X's can vanish. Suppose we require precisely n_1 of the X's to vanish, $0<n_1<n$. Applying the classical theory of linear homogeneous equations, we can determine the integer solutions of the set obtained from the n_1 vanishing X's. By (2), at least one of each of the Y's, $\cdots$, W's must now vanish, and these may be treated in the same way as the X's. The variables (3) being independent, any set obtained from the X's may be combined with any from each of the Y's, $\cdots$, W's, to give the total contribution to the solution.

2. *Equivalence.* A simple but powerful device applies the idea of reducibility to numerous interesting types of diophantine systems. Refer to (3). Let $F, G, \cdots, H$ be polynomials in the x's, y's, $\cdots$, w's, respectively, with integer coefficients (there is no gain in generality with rational coefficients, and likewise for $p', q', \cdots, r'$), and suppose that constant integers $p', q', \cdots, r'$ different from zero exist such that, *indentically*,

$$
\begin{aligned}
p'F(x_1, \cdots, x_n) &= pX_1^{a_1} \cdots X_n^{a_n}, \\
q'G(y_1, \cdots, y_m) &= qY_1^{b_1} \cdots Y_m^{b_m}, \\
\cdots \qquad & \qquad \cdots \\
r'H(z_1, \cdots, z_s) &= rW_1^{c_1} \cdots W_s^{c_s},
\end{aligned}
$$

where the capital letters denote the linear forms previously defined. Then we shall say that

$$p'F(x_1, \cdots, x_n) = q'G(y_1, \cdots, y_m) = \cdots = r'H(w_1, \cdots, w_s)$$

is *equivalent* to (2), or to any system reducible to (2). The following theorem is an immeditate consequence of the definitions.

THEOREM. *The solution of any diophantine system equivalent to a reducible system is obtainable by the algorithm of reciprocal arrays, linear congruences, and the solution of linear equations.*

3. *Examples.* Before attempting any application of the method, it is well to calculate, or at least estimate, the total number of parameters necessary and sufficient for the solution first. This number may be called the *order* of the system. It was calculated in each of the following examples. For certain quite general systems of the type (2), with the variables (1), the order has been determined by Ward in the paper cited.*

The equation

$$(x + y + z)^3 + u^3 + v^3 + w^3 = (u + v + w)^3 + x^3 + y^3 + z^3$$

illustrates §2; it is reducible, being equivalent to

$$(x + y)(y + z)(z + x) = (u + v)(v + w)(w + u).$$

The solution falls into two sets.

(i) f, g, h, k are arbitrary integers. The first set is obtained by combining any one of the following sets of values for x, y, z with any one for u, v, w:

x	y	z	u	v	w
f	$-f$	g	h	$-h$	k
f	g	$-f$	h	k	$-h$
g	f	$-f$	k	h	$-h$

(ii) $\phi_1, \cdots, \phi_9$ are parameters (as defined in §1);

$$a \equiv \phi_1\phi_4\phi_7, \quad b \equiv \phi_2\phi_5\phi_8, \quad c \equiv \phi_3\phi_6\phi_9,$$

* To see the desirability of first calculating the order, consider the system, given as an example by Ward, for one of his formulas,

$$X_1^9 = Y_1^5 = Z_1^5Z_2^4 = W_1W_2W_3W_4.$$

Although the number of algebraically independent variables of the system is only 5, the order is 46217626.

$$p \equiv \phi_1\phi_5\phi_9, \quad q \equiv \phi_2\phi_6\phi_7, \quad r \equiv \phi_3\phi_4\phi_8,$$
$$1 = (\phi_4\phi_7, \phi_5\phi_9) = (\phi_5\phi_8, \phi_6\phi_7) = (\phi_6\phi_9, \phi_4\phi_8),$$

where (i, j) denotes the G.C.D. of the integers i, j. Then

$$x = (a - b + c)/2, \qquad u = (p - q + r)/2,$$
$$y = (a + b - c)/2, \qquad v = (p + q - r)/2,$$
$$z = (-a + b + c)/2, \qquad w = (-p + q + r)/2,$$

where $\phi_1, \cdots, \phi_9$ satisfy in turn the following parity conditions, imposed to select all integers from the above rational forms of $x, \cdots, w$:

(iii) at least one parameter occurring in each of the triads a, b, c, p, q, r is even;

(iv) all 3 parameters in any one of the triads a, b, c, p, q, r are even, and the remaining 6 parameters are odd;

(v) any one of the 9 parameters is even and the remaining 8 odd.

Another completely solvable equation based on the same identity as the preceding is

$$(x + y + z)^3 = x^3 + y^3 + z^3 + w_1^{a_1} w_2^{a_2} \cdots w_s^{a_s},$$

where $a_1, \cdots, a_s$ are arbitrary constant integers >0. For, this equation is equivalent to

$$3(x + y)(y + z)(z + x) = w_1^{a_1} w_2^{a_2} \cdots w_s^{a_s},$$

which we solve by the algorithm of arrays for $x+y$, $y+z$, $z+x$, $w_1, \cdots, w_s$. From this solution x, y, z are obtained as in the preceding example.

The direct solution by the same method of

$$x^2 + y^n = z^2$$

is laborious if $n>4$; Ward's additive dual of the multiplicative method of arrays enables us to write down the rest of the solution in addition to the trivial part $y=0$, $x=\pm z$, almost by inspection. Since the equation is reducible to $y^n=uv$, with $x=(u-v)/2$, $z=(u+v)/2$, we find

$$y = \phi_1\phi_2 \cdots \phi_{n+1},$$
$$x = (\phi_1^n \phi_2^{n-1} \phi_3^{n-2} \cdots \phi_{n-1}^2 \phi_n - \phi_2\phi_3^2 \cdots \phi_{n-1}^{n-2} \phi_n^{n-1} \phi_{n+1}^n)/2,$$
$$z = (\phi_1^n \phi_2^{n-1} \phi_3^{n-2} \cdots \phi_{n-1}^2 \phi_n + \phi_2\phi_3^2 \cdots \phi_{n-1}^{n-2} \phi_n^{n-1} \phi_{n+1}^n)/2,$$

where the ϕ's are parameters. The parity restrictions on the parameters are obvious. The additive method does not (as yet) provide G.C.D. conditions on the parameters. However, if only formulas giving the solution are required, without regard to possible avoidable duplications as the parameters independently range over all integers $\neq 0$, the G.C.D. conditions can be ignored.

Numerous interesting equations which in one respect generalize the preceding examples arise from Boutin's identity,*

$$\sum \pm (\pm x_1 \pm x_2 \pm \cdots \pm x_n)^n = n!\, 2^n x_1 \cdots x_n,$$

where the sum refers to the 2^n possible combinations of signs within the parentheses, and the outer sign is the product of the inner. For example, taking $n=4$, we write down the solution of

$$\begin{aligned}&(x_1+y_1+z_1+u_1)^4+(x_2-y_2-z_2-u_2)^4\\&+(x_1-y_1-z_1+u_1)^4+(x_2+y_2+z_2-u_2)^4\\&+(x_1-y_1+z_1-u_1)^4+(x_2+y_2-z_2+u_2)^4\\&+(x_1+y_1-z_1-u_1)^4+(x_2-y_2+z_2+u_2)^4\end{aligned}$$

= the like with the suffixes 1, 2 interchanged, from the equivalent

$$x_1y_1z_1u_1 = x_2y_2u_2z_2.$$

Disregarding the trivial part in which at least one of x_i, y_i, z_i, u_i, $(i=1, 2)$, is zero, we get the rest of the solution immediately by writing down the pair of reciprocal arrays corresponding to the last equation. Thus

$$x_1 = \phi_1\phi_5\phi_9\phi_{13},\ y_1 = \phi_2\phi_6\phi_{10}\phi_{14},\ z_1 = \phi_3\phi_7\phi_{11}\phi_{15},\ u_1 = \phi_4\phi_8\phi_{12}\phi_{16},$$

$$x_2 = \phi_1\phi_6\phi_{11}\phi_{16},\ y_2 = \phi_2\phi_7\phi_{12}\phi_{13},\ z_2 = \phi_3\phi_8\phi_9\phi_{14},\ u_2 = \phi_4\phi_5\phi_{10}\phi_{15},$$

where the ϕ's are parameters, and the G.C.D. conditions are

$$(x_1, x_2) = \phi_1,\ (y_1, y_2) = \phi_2,\ (z_1, z_2) = \phi_3,\ (u_1, u_2) = \phi_4.$$

For use with cubic equations the solution of $x^3 = yzw$ is frequently required, so we state it, leaving the proof as an exercise in arrays. Neglecting the trivial part of the solution in which $x=0$, we get for the rest, in which $x \neq 0$,

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* Quoted by Dickson, *History of the Theory of Numbers*, vol. 2, p. 723. The identity, however, is due to Cauchy, who stated and proved it in a paper which has been overlooked by historians of Waring's problem.

$x = mabcfghpqr$, $y = mgh(af)^2p^3$, $z = mca(bg)^2q^3$, $w = mbf(ch)^2r^3$,

where the parameters $m, \cdots, r$ may be restricted by the G.C.D. conditions

$$1 = (a, f) = (b, g) = (c, h),$$
$$1 = (afp, bcqr) = (bgq, hfrp) = (chr, agpq).$$

The most immediate application of this is to the solution of $x^3+f(y, z, w)=0$, where $f(y, z, w)$ is any ternary cubic factorable into 3 linear, homogeneous factors whose determinant is ± 1. The solution can be written down from the above.

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A NOTE ON THE FOUR-POINT PROPERTY

BY L. M. BLUMENTHAL

1. *Introduction.* It has been pointed out by Menger that a metric space may be defined as a semi-metric space each three points of which is congruent to three points of a euclidean plane.

In a recent article, W. A. Wilson has considered a metric space that has the property that any n points of the space can be imbedded in a euclidean $(n-1)$-dimensional space.* Such a space is said to have the n-point property. An investigation of such spaces led Wilson to two important theorems. He shows (1) that if a convex, externally convex, and complete space has the four-point property, then the space has the n-point property for every integer n, and (2) that a convex, externally convex, complete, and separable space which has the four-point property is congruent with some euclidean space or with Hilbert space. It follows from the first theorem that pseudo-euclidean sets do not exist in a space satisfying the hypotheses of the theorem. This fact is of importance in connection with the problem of determining surfaces in which pseudo-euclidean n-tuples can be imbedded, for n greater than four. The second theorem, obtained by applying the first one to a well known theorem of

* W. A. Wilson, *A relation between metric and euclidean spaces*, American Journal of Mathematics, vol. 54 (1932), pp. 505–517.

Menger,* shows the great significance of the four-point property in the characterization of euclidean spaces among general semi-metric spaces.

These interesting results focus attention upon the four-point property. It is the purpose of this note to obtain a new formulation of the necessary and sufficient condition that four points of a metric space be congruent with four points of a euclidean space. While it is of interest to note that the condition turns out to be similar to the triangle axiom, the most significant feature of the new result is that it serves to relate the theory of the euclidean n-dimensional space with the spherical space of n dimensions.

2. *A New Formulation of the Condition $D_4 \geqq 0$.* If p_1, p_2, p_3, p_4 are four points of a metric space, it has been shown by Menger in the article referred to that four points of a euclidean space exist congruent to them if and only if

$$(1) \qquad D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & (p_1p_2)^2 & (p_1p_3)^2 & (p_1p_4)^2 \\ 1 & (p_2p_1)^2 & 0 & (p_2p_3)^2 & (p_2p_4)^2 \\ 1 & (p_3p_1)^2 & (p_3p_2)^2 & 0 & (p_3p_4)^2 \\ 1 & (p_4p_1)^2 & (p_4p_2)^2 & (p_4p_3)^2 & 0 \end{vmatrix} \geqq 0.$$

This condition may readily be put in the form

$$\begin{vmatrix} 2(p_1p_2)^2, & (p_1p_2)^2+(p_1p_3)^2-(p_2p_3)^2, & (p_1p_2)^2+(p_1p_4)^2-(p_2p_4)^2 \\ (p_1p_2)^2+(p_1p_3)^2-(p_2p_3)^2, & 2(p_1p_3)^2, & (p_1p_3)^2+(p_1p_4)^2-(p_3p_4)^2 \\ (p_1p_2)^2+(p_1p_4)^2-(p_2p_4)^2, & (p_1p_3)^2+(p_1p_4)^2-(p_3p_4)^2, & 2(p_1p_4)^2 \end{vmatrix} \geqq 0.$$

Now each of the four triples contained in the four points is congruent to a planar triple, since the points are, by hypothesis, in a metric space. Applying the law of cosines to the three planar triples congruent to the triples p_1, p_2, p_3; p_1, p_2, p_4; p_1, p_3, p_4, we obtain† from the preceding condition

$$8(p_1p_2)^2(p_1p_3)^2(p_1p_4)^2 \cdot \Delta \geqq 0,$$

* Menger, *New foundation of euclidean geometry*, American Journal of Mathematics, vol. 53 (1931), p. 745.

† See R. Baltzer, *Die Elemente der Mathematik*, 1885, pp. 347–348.

where

$$\Delta = \begin{vmatrix} 1 & \cos\alpha_{23} & \cos\alpha_{24} \\ \cos\alpha_{32} & 1 & \cos\alpha_{34} \\ \cos\alpha_{42} & \cos\alpha_{43} & 1 \end{vmatrix},$$

with $\alpha_{ij}=\alpha_{ji}$, $(i, j=2, 3, 4)$, where α_{ij} is the angle in the triangle formed by the points p_i', p_i', p_j' that are congruent to the points p_1, p_i, p_j, opposite the side $p_i' p_j'$. Then $0 \leqq \alpha_{ij} \leqq \pi$.

The third-order determinant Δ, whose sign is the sign of the fifth-order determinant D_4, may be written

$$\Delta = 4 \sin \tfrac{1}{2}(\alpha_{23} + \alpha_{34} + \alpha_{24}) \sin \tfrac{1}{2}(\alpha_{23} + \alpha_{34} - \alpha_{24}) \cdot \sin \tfrac{1}{2}(\alpha_{23} - \alpha_{34} + \alpha_{24}) \sin \tfrac{1}{2}(-\alpha_{23} + \alpha_{34} + \alpha_{24}).$$

Suppose, now, that Δ is positive or zero. Then the four points p_1, p_2, p_3, p_4 are congruent to four points of a euclidean space, and it is clear that the three angles α_{23}, α_{34}, α_{24} *satisfy the triangle axiom and have a sum less than or equal to* 2π. Conversely, if the angles α_{23}, α_{34}, α_{24} satisfy the triangle axiom and have a sum less than or equal to 2π, then the determinant Δ is positive or zero. This is evident, for if one angle is the sum of the other two, or if the sum of all three angles equals 2π, then one of the factors in the last expression for Δ vanishes. Also, if the sum of any two of the angles exceeds the third angle, and the sum of all three is less than 2π, it is clear that each of the factors of Δ is positive. Hence the condition $D_4 \geqq 0$ is seen to be equivalent to the condition that *the three angles* α_{23}, α_{34}, α_{24} *satisfy the triangle axiom and have a sum less than or equal to* 2π.

We say that the angles α_{23}, α_{34}, α_{24} have the point p_1 for a common vertex, and we state the above result in the following theorem.

THEOREM. *Four points of a metric space are congruent to four points of a euclidean space if and only if any three angles having a common vertex satisfy the triangle axiom with a sum not exceeding* 2π.

3. *The Spherical Space* S_2. The spherical space S_2 is formed by the points on the surface of a sphere, where the distance between two points is defined as the length of the shorter arc of the great circle joining them.

The writer has investigated the problem of characterizing sets of points congruent to points of the S_2 with radius r (r-spheric sets), as well as sets each four points of which is congruent to four points of a sphere of radius r, while the whole set is not congruent to a subset of the sphere (pseudo r-spheric sets). It has been shown in a paper not yet published in full* that the necessary and sufficient condition that three points p_1, p_2, p_3 of a semi-metric space be congruent with three points of a sphere of radius r is that (1) the distance of each two of the points is not greater than πr, and (2) the determinant Δ_3 is positive or zero, where

$$\Delta_3 = \begin{vmatrix} 1 & \cos \alpha_{12} & \cos \alpha_{13} \\ \cos \alpha_{21} & 1 & \cos \alpha_{23} \\ \cos \alpha_{31} & \cos \alpha_{32} & 1 \end{vmatrix}$$

with $\alpha_{ij} = p_i p_j / r$ radians, $(i, j = 1, 2, 3)$.

We point out that the second condition is the one investigated in the second section of this paper.

THE RICE INSTITUTE

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* L. M. Blumenthal and G. A. Garrett, *The determinantal theory of d-cyclic and pseudo d-cyclic sets of points*, this Bulletin, Abstract No. 39-1-46.

A MODULAR MANIFOLD ASSOCIATED WITH THE GENERALIZED KUMMER MANIFOLD ($p=3$)

BY E. R. OTT

1. *Introduction.* The generalized Kummer p-way manifold, K_p, is determined by equating the homogeneous coordinates of a point P in S_{2^p-1} to linearly independent theta functions of the second order and zero characteristic.* As the variables u in these functions change, the point $P(u)$ runs over the manifold K_p in S_{2^p-1}. If the variables u be increased by a half-period π, the point $P(u)$ is transformed into the point $P(u')=P(u+\pi)$. Thus the half-periods determine a group $G_{2^{2p}}$ of birational transformations of K_p into itself. Klein and Wirtinger have shown that these birational transformations are effected on K_p by the operations of a collineation group $G_{2^{2p}}$ in S_{2^p-1} under which K_p is invariant. The functions which define the position of P may be so chosen that the coefficients of the collineations of $G_{2^{2p}}$ are *numerical.* A convenient choice† is that of functions $Z_{\eta_1 \cdots \eta_p}$, $(\eta_i=0, 1)$, for which the addition of a particular half-period π_{η_i}

* The reader is referred to the following sources.

E. Bertini, *Einführung in die Projektive Geometrie Mehrdimensionaler Räume*, 1924.

A. B. Coble, *An application of finite geometry to the characteristic theory of the odd and even theta functions*, Transactions of this Society, vol. 14 (1913), pp. 241–276.

A. B. Coble, *An isomorphism between theta characteristics and the* $(2p+2)$*-point*, Annals of Mathematics, (2), vol. 17 (1916), pp. 101–112.

A. B. Coble, *Algebraic Geometry and Theta Functions*, Colloquium Publications of this Society, 1929.

R. W. H. T. Hudson, *Kummer's Quartic Surface*, 1905.

Felix Klein, *Theorie der Elliptischen Modulfunktionen*, 1890.

A. Krazer, *Lehrbuch der Thetafunktionen*, 1903.

F. Schottky, *Beziehungen zwischen den sechzehn Thetafunctionen von zwei Variabeln*, Journal für Mathematik, vol. 105 (1889), pp. 233–249.

H. Stahl, *Theorie der Abel'schen Functionen*, 1896.

W. Wirtinger, *Über eine Verallgemeinerung der Theorie der Kummer'schen Fläche and ihrer Beziehungen zu den Thetafunctionen zweier Variabeln*, Monatshefte, vol. 1 (1890).

† Coble, Colloquium, loc. cit., p. 94.

changes $Z_{\eta_1 \dots \eta_p}$ into $(-1)^{\eta_i} Z_{\eta_1 \dots \eta_p}$; and for which the addition of another particular half-period π_{η_i} interchanges the values 0, 1 of the index η_i of $Z_{\eta_1 \dots \eta_p}$, $(i=1, \cdots, p)$. The 2^{2p} points $P(u)$ for which $u=\pi$ are singular points of K_p. A particular K_p is determined when the group $G_{2^{2p}}$ of K_p, and a singular point of K_p, are given.

When $G_{2^{2p}}$ is fixed, say in the simple form just indicated, there is a family, F_p, of K_p's each of which admits this group. This family is obtained by variation of the moduli a_{ij} of the theta functions. As K_p runs through this family F_p, the 2^{2p} singular points of K_p describe a locus in S_{2p-1}, the *modular manifold* M, with which we shall be concerned for the case $p=3$. In the case $p=1$, K_p is a doubly covered S_1 with four branch points as singular points. As the modulus a of the elliptic thetas changes, these branch points run over the entire S_1. In the case $p=2$, K_p is the Kummer surface in S_3 with 16 nodal singular points. As the three moduli a_{ij} change, these nodes run over the entire S_3. In the case $p=3$, K_p is the Kummer 3-way in S_7 of order 24 with 64 four-fold singular points. In this case, however, as the *six* moduli a_{ij} change, the singular points in S_7 run over a manifold of dimension six and order 16, M_6^{16}. The purpose of this article is to discuss this manifold M_6^{16} which appears from the transcendental view-point, with respect to its projective and group-theoretic properties. We use without further explanation the standard notations of the theta function theory. The equation of M_6^{16} is obtained in §2. In §§3–4, the multiplicities of certain loci on M_6^{16} and on its sections are determined. In §5 a projective determination of M_6^{16} by means of these multiplicities is indicated.

2. *Determination of the Equation of* M_6^{16}. The equation of M_6^{16} may be obtained from one of the relations existing between products of the zero values of the even thetas. One such is

$$\vartheta_{lmn7}\vartheta_{lmn8}\vartheta_{iln7}\vartheta_{iln8} \pm \vartheta_{jmn7}\vartheta_{jmn8}\vartheta_{ijn7}\vartheta_{ijn8} \pm \vartheta_{kmn7}\vartheta_{kmn8}\vartheta_{ikn7}\vartheta_{ikn8} = 0.$$

This may be written, in a particular case, as

$$\sum_{\alpha=1}^{3} \pm (c^2_{\alpha 457} c^2_{\alpha 458} c^2_{\alpha 467} c^2_{\alpha 468})^{1/2} = 0, \tag{1}$$

where $\vartheta_{ijkl}(0)=c_{ijkl}$. To obtain the expression for this in terms of the modular functions $z_{ijk}=z_{ijk}(0)$, it is convenient to write the relation as $(a)^{1/2}+(b)^{1/2}+(c)^{1/2}=0$. The rational form is $c^2-2c(a+b)+(a-b)^2=0$. In expressing the c^2_{ijkl} in terms of the z_{ijk}'s it is convenient to set $z_{000}=z$, and the remaining $z_{ijk}=z_1, z_2, \cdots, z_7$, where the $z_1, z_2, \cdots, z_7$ are thought of as attached to the seven points of a finite planar geometry, mod 2, say $PG(2, 2)$, in particular z_{ijk} attaching to the point with coordinates $i, j, k=0, 1$. Then the a, b, c, above, become

$$\begin{aligned}
a = {} & [(z^2-z_6^2)+(z_1^2-z_2^2)+(z_3^2-z_7^2)+(z_5^2-z_4^2)] \\
& \cdot[(z^2-z_6^2)-(z_1^2-z_2^2)-(z_3^2-z_7^2)+(z_5^2-z_4^2)] \\
& \cdot[(z^2-z_6^2)+(z_1^2-z_2^2)-(z_3^2-z_7^2)-(z_5^2-z_4^2)] \\
& \cdot[(z^2-z_6^2)-(z_1^2-z_2^2)+(z_3^2-z_7^2)-(z_5^2-z_4^2)], \\
b = {} & [(z^2+z_6^2)-(z_1^2+z_2^2)+(z_3^2+z_7^2)-(z_5^2-z_4^2)] \\
& \cdot[(z^2+z_6^2)+(z_1^2+z_2^2)-(z_3^2+z_7^2)-(z_5^2+z_4^2)] \\
& \cdot[(z^2+z_6^2)-(z_1^2+z_2^2)-(z_3^2+z_7^2)+(z_5^2-z_4^2)] \\
& \cdot[(z^2+z_6^2)+(z_1^2+z_2^2)+(z_3^2+z_7^2)+(z_5^2+z_4^2)], \\
c = {} & [2zz_6+2z_1z_2+2z_3z_7+2z_4z_5]\cdot[2zz_6+2z_1z_2-2z_3z_7-2z_4z_5] \\
& \cdot[2zz_6-2z_1z_2+2z_3z_7-2z_4z_5]\cdot[2zz_6-2z_1z_2-2z_3z_7+2z_4z_5].
\end{aligned}$$

The rationalized form then yields the following equation for M_6^{16}:

$$\begin{aligned}
M_6^{16} = {} & zz_1z_2z_3z_4z_5z_6z_7\Big\{-2z^8+4z^4\sum_7 z_1^4-16z^2\sum_7 z_1^2z_2^2z_6^2 \\
& -2\sum_7 z_1^8-16\sum_7 z_3^2z_4^2z_5^2z_7^2+4\sum_{21} z_1^4z_2^4\Big\} \\
& +z^8\Big(\sum_7 z_3^2z_4^2z_5^2z_7^2\Big)+z^6\Big(-\sum_{7\cdot 4} z_1^4z_2^2z_3^2z_4^2\Big) \\
& +z^4\Big(-\sum_{7\cdot 4} z_3^6z_4^2z_5^2z_7^2+2\sum_{7\cdot 3} z_6^4z_3^2z_4^2z_5^2z_7^2+\sum_{28} z_1^4z_2^4z_3^4\Big) \\
& +z^2\Big(\sum_{7\cdot 4} z_6^8\cdot z_1^2z_3^2z_5^2-\sum_{7\cdot 4\cdot 3} z_6^4z_1^6z_3^2z_5^2+2\sum_{7\cdot 6} z_6^4z_7^4z_1^2z_3^2z_5^2
\end{aligned} \tag{2}$$

$$+ 72z_1^2z_2^2z_3^2z_4^2z_5^2z_6^2z_7^2\Big) + \Big(\sum_{7\cdot 3} z_6^8 \cdot z_3^2z_1^2z_5^2z_7^2 - \sum_{7\cdot 4\cdot 3} z_6^6z_2^2z_1^4z_3^2z_5^2$$

$$+ \sum_{7\cdot 4} z_6^4 \cdot z_1^4z_3^4z_5^4 + 2\sum_{21} z_6^4z_2^4 \cdot z_3^2z_4^2z_5^2z_7^2\Big) = 0.$$

This final form of the equation of M_6^{16} contains first the odd and then the even powers of z in descending order. The symbol $\sum_7 z_1^2z_2^2z_6^2$ represents the sum of seven terms whose three subscripts lie on a line in the finite geometry,* $PG(2, 2)$. Other symbols in (2) refer to the seven quadrilaterals, seven points with each of four outside lines, seven points with each of three outside quadrangles, twenty-eight triangles, and twenty-one pairs of intersecting lines. Thus the modular locus is of order 16, with a 7-fold point at each reference point with a tangent septimic cone made up of the seven reference S_6's on the 7-fold point, and with similar behavior at the conjugates of these under the group of M_6^{16}.

This modular manifold, M_6^{16}, is invariant† under a group G of order $8!\cdot 36\cdot 64$ generated by elements $J_{abc,lmn}$. The situation is given by the following statement.

THEOREM 1. *M_6^{16} is invariant under a group G of order $8!\cdot 36\cdot 64$. This group G is the largest collineation group which contains the collineation group G_{64} of K_3^{24} as an invariant subgroup.*

Corresponding to the fact that the seven points of a finite plane (mod 2) may be permuted in 168 ways without destroying the linearity of triads, the equation of M_6^{16} is invariant under a permutation group of $z_1, \cdots, z_7$ of order 168, a subgroup of G. There exists a larger permutation subgroup $G_{8\cdot 168}$ on $z, z_1, \cdots, z_7$. Under the symmetry imposed by this larger subgroup in the equation (2) of M_6^{16} is reduced to the form

$$M_6^{16} = \Big\{\prod_8(z)\Big\}\Big\{-2\sum_8 z^8 + 4\sum_{28} z^4z_1^4 - 16\sum_{14} z^2z_1^2z_2^2z_6^2\Big\}$$

$$(3) \quad + \sum_{7\cdot 8} z^8z_3^2z_4^2z_5^2z_7^2 - \sum_{7\cdot 4\cdot 8} z^6z_1^4z_2^2z_3^2z_4^2 + 2\sum_{7\cdot 12} z^4z_6^4z_3^2z_4^2z_5^2z_7^2$$

$$+ \sum_{7\cdot 8} z^4z_1^4z_2^4z_3^4 + 72\Big\{\prod_8(z)\Big\}^2 = 0.$$

* Coble, Colloquium, loc. cit., p. 105.

† Coble, Colloquium, loc. cit., pp. 100–101, and p. 107.

3. *Multiple Loci of* M_6^{16}. A. *7-fold points.* The reference octahedron is one of a set of 135 octahedra all of which are conjugate under G. A particular octahedron is associated with a Göpel system of 7 mutually syzygetic half periods. No two of these octahedra have a vertex in common, whence the number of 7-fold points of M_6^{16} is $8 \cdot 135 = 1080$.

B. *4-fold* S_3*'s.* To each half-period of the theta functions there is associated an element J of period four whose square I is in G_{64}. As an involutorial element in G_{64}, I has a locus of fixed points made up of the two skew S_3's. Thus there is a correspondence between the 63 half-periods and 63 pairs of skew S_3's. A typical pair of S_3's is $z = z_1 = z_2 = z_6 = 0$, $z_3 = z_4 = z_5 = z_7 = 0$. From the form of (2), we have the following theorem.

THEOREM 2. *Each of the* 126 S_3*'s determined by the* 63 *half-periods lies on* M_6^{16} *and is a* 4*-fold locus of it.*

The sum of two half-periods is a third. Such a linearly related triad of half-periods will contain pairs which are either (a) syzygetic or (b) azygetic. We examine the intersections of the three pairs of S_3's determined by such syzygetic or azygetic triads. The number of such triads is 315 or 336 as the case may be.*

In case (a) as it appears for $p = 2$, the syzygetic triad of half-periods determines three pairs of lines which are the three pairs of opposite edges of a 4-point in S_3. For $p = 3$, however, they determine three pairs of S_3's which are the three pairs of opposite S_3's on a 4-line λ in S_7. Thus each line λ is on three S_3's. It is a line joining two vertices of one of the 135 reference octahedra and each line λ is on three pairs of such vertices. These 1260 lines λ are an extension of the 60 points of Klein's 60_{15} configuration ($p = 2$). We may state the following result.

THEOREM 3. *The* 126 S_3*'s determined by the* 63 *half-periods intersect by threes in* 1260 *lines* λ *which are* 6*-fold lines of* M_6^{16}. *These lines constitute one extension of Klein's* 60_{15} *configuration* ($p = 2$), *the other being the* 135 *reference octahedra. The* 30 *lines* λ *in each* S_3 *form the edges of a* 60_{15} *configuration* ($p = 2$).

* Coble, Colloquium, loc. cit., §33.

Thus a plane through three vertices of a reference octahedron in S_7 cuts the manifold in an 18-ic, and therefore lies on the manifold. Any one of these planes is a 4-fold locus on M_6^{16}.

In case (b), and for $p=2$, the azygetic triad of linearly related half-periods determines three pairs of lines which are generators of one ruling of the quadric in S_3 which is determined by that even theta function which has the three half-periods as zeros. For $p=3$, they determine three pairs of S_3's any two of which are skew to each other. These six S_3's are S_3-generators of one ruling* for each of the quadrics in S_7 determined by each of the six even theta functions which has the three half-periods as zeros.

4. *Linear Sections of* M_6^{16}. We examine only those linear sections of M_6^{16} which are most effective for its projective determination. We have already noted that the S_3 of the type $z=z_1=z_2=z_6=0$ lies on M_6^{16}. On the other hand, from the equation of M_6^{16}, we have the following theorem.

THEOREM 4. *An* S_3 *of the type* $z=z_1=z_2=z_3=0$ *cuts* M_6^{16} *in the four faces of a tetrahedron each repeated four times.*

We shall also need the sections M_4^{16} of M_6^{16} by S_5's of the type $z=z_1=0$. Each of the 135 reference octahedra determines 28 of these S_5's but each S_5 occurs in three octahedra, whence there are $45\cdot 28$ of these S_5's and each is invariant under a subgroup $G_{3\cdot 48\cdot 2^9}$ of G. The factor 3 of this order is due to the three octahedra which contain the S_5; the factor 48 is the order of the subgroup of permutations of $z, \cdots, z_7$ which leaves the pair z, z_1 unaltered; and the factor 2^9 represents the multiplicative subgroup of $z, \cdots, z_7$ which appears in G.

Such an S_5 is cut by the 63 pairs of S_3's of Theorem 2 in pairs of linear spaces of the following character: (α) 3 consisting of an S_3 and an S_1; (β) 12 consisting of pairs of S_2's; (γ) 48 consisting of pairs of S_1's. According to Theorem 2 these are four-fold loci on M_4^{16}.

THEOREM 5. *An* M_4^{16} *cut out on* M_6^{16} *by an* S_5 *of type* $z=z_1=0$ *has the loci* (α), (β), (γ) *and the faces described in Theorem* 4 *as four-fold loci.*

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* Bertini, loc. cit., p. 143.

The S_1's of (α) are actually 8-fold loci on M_4^{16}. It is now relatively easy to prove the following theorem.

THEOREM 6. *M_4^{16} is the manifold of lowest order in S_5 invariant under $G_{18\cdot 2\cdot 2^9}$ with the multiplicities described in Theorem* 5.

For its invariance under the multiplicative group of order 2^9 rules out all terms whose exponents do not satisfy certain congruences. Its invariance under the permutation group of order 48 mentioned above necessitates the equality of various sets of coefficients. The remaining indeterminations are then easily removed by applying the multiplicity conditions of Theorems 4, 5.

5. *Determination of a Unique Manifold of Lowest Order with Multiplicities of* §3, *Invariant under the Group G of Order* $8!\cdot 36\cdot 64$. The original form of the equation of M_6^{16} was obtained from transcendental considerations. But the invariance of M_6^{16} under the multiplicative group of order 2^9, $(J_{abc,lmn})$, is sufficient to determine the type of terms which may appear in its equation. This is accomplished by the solution of congruences. Its invariance under the permutation group $G_{8\cdot 168}$ of $z, \cdots, z_7$ reduces the number of unknown coefficients to 8. These coefficients may now be determined by applying Theorem 6 to the S_5's defined by the reference octahedron. Thus a characteristic projective property of the manifold, M_6^{16}, originally defined by considerations in which function theory is essential, has been obtained. We may then make the following statement.

THEOREM 7. *M_6^{16} is the manifold of lowest order in S_7 invariant under the group G of order* $8!\cdot 36\cdot 64$ *with the multiplicities of* §3.

THE UNIVERSITY OF ARKANSAS

INVOLUTORIAL LINE TRANSFORMATIONS DETERMINED BY CREMONA PLANE INVOLUTIONS*

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1. *Introduction.* The author has discussed† an involutorial line transformation effected by considering a harmonic homology in each of two planes. If A, B be the points in which an arbitrary line (y) meets the planes α, β, and if A', B' be their images by the homologies I_α, I_β, respectively, then $(x) \equiv A'B'$ is the transform of (y). It is the purpose of the present paper to consider the line transformations similarly determined by Cremona involutorial transformations in each of two planes. All combinations of the four fundamental types: Homology; Jonquières; Geiser; and Bertini will be considered. The orders of the transformations, the invariant loci, the singular elements and the transforms of certain elementary forms are discussed.

2. *Homology-Jonquières.* In the plane α consider a harmonic homology I_α, center at O_1 and axis Δ_α. In the plane β consider the perspective Jonquières involution I_β, of order n, with basis point P_1 of multiplicity $(n-1)$ and basis points $P_2, \cdots, P_{2n-1}$ each simple, and with invariant curve $\Delta_\beta: P_1^{n-2} P_2^1 \cdots P_{2n-1}^1$ of order n and genus $(n-2)$.

An arbitrary line (y) meets α in a point A whose coordinates are linear in the Plücker coordinates y_i of (y) and meets β in a point B whose coordinates are also linear in y_i. The image A' of A by I_α has coordinates also linear in y_i but the image B' of B by I_β has coordinates which are functions of degree n in y_i. Hence $(x) \equiv A'B'$ has Plücker coordinates of degree $(n+1)$ in y_i. Thus, the transformation

$$x_i = \phi_i(y) \tag{1}$$

is of order $(n+1)$. The invariant lines of (1) form a congruence (n, n) composed of the lines meeting Δ_α, Δ_β; and in addition there is a cone of order n, vertex O_1, base curve Δ_β.

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† *Some involutorial line transformations*, this Bulletin, vol. 39 (1933), pp. 149–154.

If (y) meets β in P_1 and α in A, then $(y)\sim$ a cone of order $(n-1)$, vertex A', base curve the curve of order $(n-1)$ into which P_1 is transformed by I_β. If (y) meets β in $P_i (i\neq 1)$ and α in A, then $(y)\sim$ the pencil whose vertex is A' and in the plane $A'P_1P_i$.

The image by I_α of the line $c\equiv\alpha\beta$ is a line c_α and by I_β is a curve c_β of order n. The points C_α of c_α and C_β of c_β are projective with the points C of c. Hence the line joining any two corresponding points C_α, C_β is transformed by (1) into the bundle whose vertex is C. These lines $C_\alpha C_\beta$ form a regulus $\{c\}^{n+1}$ of order $(n+1)$, every generator of which is singular.

A line t_α in α meets β in a point C whose image by I_β is C_β. The image of t_α by I_α is a line t_α'. Hence $(y)\equiv t_\alpha\sim$ a pencil, vertex C_β, plane $C_\beta t_\alpha'$. If t_α pass through O_1, then $t_\alpha'\equiv t_\alpha$ and the plane of the pencil is $C_\beta t_\alpha$. If t_α be the axis Δ_α, then $t_\alpha'\equiv\Delta_\alpha$ and the plane of the pencil is $C_\beta\Delta_\alpha$.

A line t_β in β meets α in a point C whose image by I_α is C_α. The image of t_β by I_β is a curve ρ_β of order n. Hence $(y)\equiv t_\beta\sim$ a cone of order n, vertex C_α, base curve ρ_β. If t_β pass through P_1, then ρ_β is a line, and indeed the line t_β. Hence the conjugate is no longer a cone but a pencil, vertex C_α, plane $C_\alpha t_\beta$. If t_β pass through $P_i (i\neq 1)$, then ρ_β is of order $(n-1)$ and the conjugate cone is of order $(n-1)$. If t_β be the line P_1P_i, then $(y)\equiv t_\beta\sim C_\alpha P_i$.

As t_α describes the pencil (C, α) each conjugate pencil has its vertex at C_β and its plane passes through C_α. As C describes c, C_β describes c_β, and hence the plane field $(\alpha)\sim$ the special complex of order n with c_β as directrix curve.

As t_β describes the pencil (C, β), then each conjugate cone has its vertex at C_α and has $C_\alpha C_\beta$ as a generator. As C describes c, C_α describes c_α, and hence the plane field $(\beta)\sim$ the special linear complex $|c_\alpha|$.

Each line t_α of a pencil (T, α) is transformed by (1) into a pencil whose vertex is on c_β and whose plane passes through t_α' which belongs to the pencil (T', α), T' being the image by I_α of T. These ∞^1 pencils form a congruence $(n+1, n)$. If a line (x) meet c_β in a point C_β and also meet the t_α' which corresponds to C_β, then (x) belongs to the conjugate congruence. Through an arbitrary point of space a line (x) belonging to the conjugate congruence has coordinates which are of degree $(n+1)$ in the parameter λ of a line of (T, α). Hence the order of the con-

gruence is $(n+1)$. An arbitrary plane meets c_β in n points and meets each corresponding t_α' in one point. Hence the class is n.

Likewise a pencil $(T, \beta) \sim$ a congruence $(n+1, n)$. However, if T lie at P_1, the order and class are both reduced so that the congruence is (2, 1). If T lie at $P_i (i \neq 1)$, then the congruence is $(n, n-1)$.

An arbitrary pencil $(T, \tau) \sim$ a regulus R of order $(n+1)$, the generators of which are the joins of corresponding points on the straight line image by I_α of $\tau\alpha$ and the curve of order n, image by I_β of $\tau\beta$. If τ pass through P_1, then R is of order 2; through $P_i (i \neq 1)$, of order n; through $P_iP_j (i, j \neq 1)$, of order $(n-1)$; through P_1P_i, the conjugate is a pencil with vertex P_i.

An arbitrary plane field of lines $(\tau) \sim$ a congruence (n, n) composed of lines meeting a line and a plane curve of order n not meeting the line. If τ pass through P_1, the congruence is (1, 1); through $P_i (i \neq 1)$, $(n-1, n-1)$; through $P_iP_j (i, j \neq 1)$, $(n-2, n-2)$; through P_1P_i, the conjugate is no longer a congruence but a pencil, vertex P_i.

An arbitrary bundle $(T) \sim$ a congruence $(3n, n)$. From an arbitrary point of space, the points of the planes α, β form two projective fields. There are three coincidences in a section of such a projection. Hence the parameters λ, μ of a line of (T) appear to degree $3n$ in defining a line (x) of the conjugate congruence through an arbitrary point of space. In an arbitrary plane of space lie n lines of the congruence.

A bilinear congruence $[|d_1|, |d_2|] \sim$ a congruence $(4n, 2n)$. The transformation (1) is involutorial. Hence the conjugate congruence of the (1, 1) will be of order equal to the number of lines common to the (1, 1) and the conjugate of an arbitrary bundle, which is $4n$. Likewise, the class will be the number of lines common to the (1, 1) and the conjugate of an arbitrary plane field.

A linear complex is transformed by (1) into a complex of order $(n+1)$ since this is the order of the transformation.

3. *Homology-Geiser.* Consider I_α in α as before, and in β, $I_\beta: P_1^3 \cdots P_7^3$, with invariant curve Δ_β of order 6 having double-points at $P_1, \cdots, P_7$. The order of the transformation is 9; the invariant lines form a congruence (6, 6) and a cone of order 6, vertex O_1, base curve Δ_β; the singular elements are, as

before, the lines of the bundles whose vertices are F-points of I_β, a regulus $\{c\}^9$ of order 9 and the plane fields (α), (β). To any line of any bundle corresponds a cone of order 3; to any generator of $\{c\}^9$, a bundle, vertex on c; to (α) a special complex of order 8, and to (β) a special linear complex. If t_β pass through one or two points P_i, its conjugate cone is of order 5 or 2. If the vertex T of a pencil (T, β) be at P_i, the conjugate congruence of the pencil is (6, 5); otherwise (T, α) or $(T, \beta) \sim$ a congruence (9, 8).

If the plane τ of an arbitrary pencil (T, τ) pass through one or two points P_i, the conjugate regulus is of order 6 or 3. Otherwise $(T, \tau) \sim$ a regulus of order 9.

If the plane τ pass through j points $P_i (j=0, 1, 2)$, $(\tau) \sim$ a congruence $(8-3j, 8-3j)$.

An arbitrary bundle $(T) \sim$ a congruence (24, 8).

A congruence (1, 1) $\sim$ a congruence (32, 16), and a linear complex $\sim$ a complex of order 9.

4. *Homology-Bertini.* Consider I_α as before and $I_\beta: P_1^6 \cdots P_8^6$, Δ_β being of order 9 with triple-points at P_i. The order of the transformation is 18; the invariant lines form a congruence (9, 9) and a cone of order 9; the singular elements are the lines of the bundles (P_i), a regulus $\{c\}^{18}$ of order 18, and the plane fields (α), (β). Any line of any $(P_i) \sim$ a cone of order 6, each generator of $\{c\}^{18} \sim$ a bundle with vertex on c, $(\alpha) \sim$ a special complex of order 17, and $(\beta) \sim$ a special linear complex. The conjugate cone of a line t_β has its order $17-6j$, where j is the number of points P_i on t_β. The conjugate congruence of a pencil (T, α) or (T, β) is (18, 17) unless T lie at some P_i, when the congruence is (12, 11).

If τ pass through j points $P_i (j=0, 1, 2)$, then the pencil $(T, \tau) \sim$ a regulus of order $18-6j$ and the plane field $(\tau) \sim$ a congruence $(17-6j, 17-6j)$.

A bundle $(T) \sim$ a congruence (51, 17).

A congruence (1, 1) $\sim$ a congruence (68, 34), and the transform of a linear complex is a complex of order 18.

5. *Jonquières-Jonquières.* When we consider two perspective Jonquières involutions, I_α of order m, center O_1, and I_β of order n, center P_1, the order of the transformation is $(m+n)$. The invariant lines form a congruence (mn, mn) since now Δ_α and Δ_β

are of orders m, n, respectively, and each passes simply through the simple F-points in its respective plane and multiply through the center. In addition to the singular regulus $\{c\}^{m+n}$ of order $(m+n)$ whose generators are transformed into bundles with vertices on c, and the singular plane fields (α), (β) whose conjugates are special complexes of orders n, m, respectively, and the singular bundles whose vertices are at F-points and each of whose lines is transformed into a cone whose order is the multiplicity of the F-point in I_α or I_β, there are $(2m-1)(2n-1)$ singular lines O_iP_j whose conjugates are congruences. If $i, j \neq 1$, each congruence is $(1, 1)$; if $i=1$, $j\neq 1$, each congruence is (m, m); if $i\neq 1$, $j=1$, each is (n, n); if $i=j=1$, the conjugate congruence is $([m-1][n-1], [m-1][n-1])$. A line $(y) \equiv t_\alpha \sim$ a cone of order $(n-j)$, where j is the sum of the multiplicities of F-points O_i on t_α, and $(y) \equiv t_\beta \sim$ a cone of order $(m-i)$, where i is the sum of the multiplicities of F-points P_l on t_β. A pencil $(T, \beta) \sim$ a congruence $(m+n-i, m[n-i])$ where i is the multiplicity of T as an F-point of I_β and $(T, \alpha) \sim$ a congruence $(m-j+n, [m-j]n)$, where j is the multiplicity of T as an F-point of I_α.

An arbitrary pencil $(T, \tau) \sim$ a regulus R of order $(m-i+n-j)$, where i is the sum of the multiplicities of F-points O_k and j the sum of the multiplicities of F-points P_l lying in τ. If $i=m$ or $j=n$, then R is a cone. Both $i=m$ and $j=n$ will not occur if I_α, I_β be taken arbitrarily.

A plane field $(\tau) \sim$ a congruence whose order and class are both $(m-i)(n-j)$, where i, j are as defined immediately above. In the event $m-i=0$ or $n-j=0$, the conjugate of (τ) is a cone whose order is the factor which does not vanish.

An arbitrary bundle $(T) \sim$ a congruence $(3mn, mn)$.

A congruence $(1, 1) \sim$ a congruence $(4mn, 2mn)$, and a linear complex has for conjugate a complex of order $(m+n)$.

6. *Jonquières-Geiser.* Take I_α as in §5 and I_β as in §3. The transformation is of order $(m+8)$; the invariant lines form a congruence $(6m, 6m)$; the singular elements are the bundles (O_i), (P_j), the $7(2m-1)$ lines O_iP_j, the plane fields (α), (β), and the singular regulus $\{c\}^{m+8}$ defined as in each previous case. Each line of $(O_i) \sim$ a cone of order $(m-k)$, where k is the multiplicity of O_i in I_α, and each line of $(P_j) \sim$ a cone of order 3;

$(y) \equiv O_i P_j \sim$ a congruence whose order and class are both $5(m-k)$, where k is as defined just above; $(\alpha) \sim$ a special complex of order 8 and $(\beta) \sim$ a special complex of order m; each generator of $\{c\}^{m+8} \sim$ a bundle whose vertex is on c. Finally $(y) \equiv t_\alpha \sim$ a cone of order $(m-i)$, where i is the sum of the multiplicities of F-points on t_α, and $(y) \equiv t_\beta \sim$ a cone of order $(8-j)$, where j is the sum of the multiplicities of F-points on t_β.

A pencil $(T, \alpha) \sim$ a congruence $(m-i+8, [m-i]8)$, where i is the multiplicity of T as an F-point of I_α, and a pencil $(T, \beta) \sim$ a congruence $(m+8-j, m[8-j])$, where j is the multiplicity of T as an F-point of I_β.

An arbitrary pencil $(T, \tau) \sim$ a regulus R whose order is $(m-i+8-j)$, where i is the sum of multiplicities of F-points O_k and j the sum of multiplicities of F-points P_l lying in τ. If I_α, I_β be taken arbitrarily, no more than 3 points O_k, P_l lie on τ. If $i=m$, R is a cone of order $(8-j)$.

A plane field $(\tau) \sim$ a congruence whose order and class are both $(m-i)(8-j)$, where i, j are as defined in the preceding paragraph. If $i=m$, the conjugate of (τ) is a cone of order $(8-j)$.

A bundle $(T) \sim$ a congruence $(24m, 8m)$.

A congruence $(1, 1) \sim$ a congruence $(32m, 16m)$ and a linear complex is transformed into complex of order $(m+8)$.

7. *Jonquières-Bertini.* Take I_α as in §5 and I_β as in §4. The transformation is of order $(m+17)$, and all of the results follow by replacing 8 by 17 in the preceding section, except that a bundle $(T) \sim$ a congruence $(51m, 17m)$ and a congruence $(1, 1) \sim$ a congruence $(68m, 34m)$.

8. *Geiser-Geiser.* Given $I_\alpha : O_1^3 \cdots O_7^3$ and I_β as in §3. The transformation is of order 16; the invariant lines form a congruence (36, 36); the singular elements are (O_i), (P_j), $O_i P_j$, $\{c\}^{16}$, (α), (β), and their conjugates are easily found by the methods outlined above. The conjugates of the elementary forms (T, τ), (τ), (T), (1, 1) and linear complex are also readily obtained.

9. *Geiser-Bertini.* Consider I_α as in §8 and I_β as in §4. The transformation is of order 25; the invariant lines form a congruence (54, 54); the singular elements are (O_i), (P_j), $O_i P_j$, $\{c\}^{25}$, (α), (β), and their conjugates, and those of the elemen-

tary forms follow immediately from the method we have used throughout this paper.

10. *Bertini-Bertini.* Given $I_\alpha: O_1^6 \cdots O_8^6$ and I_β as in §4. The order of the transformation is 34, and by repetition of what has been done above we can discuss this transformation completely.

If fundamental elements of one or both Cremona plane involutions lie on the line $c \equiv \alpha\beta$ the preceding results must be modified. The details are not difficult in each particular case, but the large number of possible forms cannot be considered here.

CORNELL UNIVERSITY

A SEPARATION THEOREM*

BY W. A. WILSON

Various writers on topology have had occasion in the course of their work to prove lemmas of the following general nature. Given sets A and B lying in a connected space Z, it is possible to express Z as the union of two continua M and N such that $N \cdot (A - A \cdot B) = M \cdot (B - A \cdot B) = 0$, provided that A, B, and Z satisfy the proper conditions. The last of these to come to the writer's attention are two theorems by Vietoris and one by the author of this note.† Such theorems are of course generalizations of Tietze's separation axioms‡ and it might prove profitable to work out systematically the possibilities along this line.

In some of the generalizations mentioned it is shown that, if Z is locally connected, a decomposition $Z = M + N$, where M and N are also locally connected, is possible, but, as far as the writer knows, the following theorem, which shows a certain kind of local connectivity for $M \cdot N$ as well as for M and N, is new.

THEOREM. *Let A and B be sub-continua of the locally connected compact metric space Z and let $A \cdot B$ be totally disconnected or void.*

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* Presented to the Society, February 25, 1933.

† L. Vietoris, *Über den höheren Zusammenhang*, Fundamenta Mathematicae, vol. 19, pp. 271–272; W. A. Wilson, *On unicoherency about a simple closed curve*, American Journal of Mathematics, vol. 55, p. 141.

‡ See F. Hausdorff, *Mengenlehre*, p. 229.

Then there are two locally connected continua M and N such that $Z=M+N$, $M\cdot(B-A\cdot B)=0=N\cdot(A-A\cdot B)$, $A\cdot B\subset M\cdot N$, and each component of $M\cdot N$ is locally connected.

PROOF. Set $\alpha=A-A\cdot B$ and $\beta=B-A\cdot B$. Since A and B are continua and $A\cdot B$ is totally disconnected, $A=\bar{\alpha}$ and $B=\bar{\beta}$. As α and β are mutually separated sets, there are closed sets F and G such that $Z=F+G$, $\alpha\cdot G=0$, and $\beta\cdot F=0$. Let R be the union of the components of $Z-G$ which contain points of α. Then $A\subset\overline{R}$ and so $\overline{R}$ is a continuum. Let S be the union of the components of $Z-\overline{R}$ which contain points of β, and let T be the union of the other components. As in the case of $\overline{R}$, we see that $B\subset\overline{S}$ and $\overline{S}$ is a continuum. Set $H=\overline{R}+T$ and $K=\overline{S}$. Clearly H is a continuum, $Z=H+K$, $\beta\cdot H=0$, $\alpha\cdot K=0$, and $A\cdot B\subset H\cdot K$.

The continua H and K are not necessarily locally connected at points of $H\cdot K$, although they are locally connected at all other points. Set $J=H\cdot K-A\cdot B$. The set J contains no point of $A+B$ and it is the union of a sequence of closed sets $\{J_i\}$ defined as follows. Take a sequence of positive numbers $\{\eta_i\}$ such that $2\eta_{i+1}<\eta_i$, let J_1 be the set of points of J whose respective distances from $A\cdot B$ are not less than η_1, and in general let J_i be the set of points of J whose respective distances from $A\cdot B$ are not more than η_{i-1} and not less than η_i. (Some of these may be void.) We then choose a descending sequence of positive numbers $\{\sigma_i\}$ converging to zero such that each σ_i is less than the distance between J_i and $A+B$. If $\epsilon_i<\sigma_i/3$, there is a finite set of locally connected continua whose union we call L_i such that the points of J_i are inner points of L_i (relative to Z) and $L_i\subset V\epsilon_i(J_i)$.* If also ϵ_i is small enough, no two sets L_i and L_j will have common points unless $|i-j|=1$.

Let L be the union of the sets $\{L_i\}$. If a is a limiting point of L not contained in L, it is in no L_i and is the limit of a sequence of points $\{a_j\}$, where each a_j lies in L_j and j runs over a partial sequence of the numbers $i=1, 2, 3, \cdots$. Since $\epsilon_i\to 0$, $\eta_i\to 0$, and $L_i\subset V\epsilon_i(J_i)$, it follows that a lies in $A\cdot B$. Thus $\overline{L}\subset L+A\cdot B$. Note also that $L\cdot(A+B)=0$.

Let $P=L+A\cdot B$ and C be a component of P. If C has but one point, it is locally connected in a trivial sense. In the other

* By this notation is meant the set of points (of Z) whose respective distances from J_i are less than ϵ_i.

event C is the sum of a sub-set of L and a sub-set of $A \cdot B$. If a is a point of C not in $A \cdot B$, a lies in one L_i only or perhaps in $L_i + L_{i+1}$. In either event there is a $\delta > 0$ for which $P \cdot V_\delta(a)$ lies in a locally connected sub-continuum of $L_i + L_{i+1}$, which in turn is contained in C. But then C is locally connected at *every* point, since $C \cdot A \cdot B$ is totally disconnected and no continuum is locally connected at all points except those of a totally disconnected set. Thus each component of P is locally connected.

Now set $M = H + P$ and $N = K + P$. Clearly $M \cdot N = P$, $M \cdot (B - A \cdot B) = N \cdot (A - A \cdot B) = 0$, and $A \cdot B \subset M \cdot N$. It is obvious that M is locally connected at the points of $M - P$. For any other point x we have $M \cdot V_\epsilon(x) = H \cdot V_\epsilon(x) + P \cdot V_\epsilon(x)$. If x lies in $H - H \cdot K$ and ϵ is small enough, $V_\epsilon(x) \subset H - H \cdot K$, and therefore M is locally connected at x. If x lies in $K - H \cdot K$, then $M \cdot V_\epsilon(x) = P \cdot V_\epsilon(x)$ for ϵ small enough. In consequence of the discussion in the previous paragraph we can also take ϵ so small that $P \cdot V_\epsilon(x)$ lies in some one component C of P and therefore M is locally connected at x on account of the local connectivity of C at x. If x lies in $J = H \cdot K - A \cdot B$, then x is an inner point of some one of the locally connected continua whose union is L, whence M is locally connected at x. Thus M is locally connected at all points except possibly those of the totally disconnected set $A \cdot B$; it is therefore everywhere locally connected. Likewise N is everywhere locally connected and the theorem is proved.

In conclusion it should perhaps be noted that the number of components of $M - N$ depends upon the nature of Z; for example, if Z is further restricted to being unicoherent, then $M \cdot N$ is a locally connected continuum.

YALE UNIVERSITY

ON THE GENERALIZED VANDERMONDE DETERMINANT AND SYMMETRIC FUNCTIONS

BY ARNOLD DRESDEN

1. *Introduction.* E. R. Heineman has published* a direct method for expressing an arbitrary symmetric polynomial in n variables $a_1, a_2, \cdots, a_n$ in terms of the elementary symmetric functions $E_1, E_2, \cdots, E_n$ of these variables. His method was based on a combination of a formula for the general Vandermonde determinant developed by him, with a theorem of Muir.† In the present note we develop in a very elementary way a simple formula for the quotient of a general Vandermonde determinant by the simple alternant, in terms of the "homogeneous product sums" of weight s.‡ These homogeneous product sums can be expressed explicitly in terms of the elementary symmetric functions.§ The formula which is obtained gives, therefore, by the use of Muir's theorem, another straightforward method for the calculation of an arbitrary symmetric polynomial.

2. *A Theorem on Determinants.* It is an exercise in elementary algebra to prove that the determinants

$$\begin{vmatrix} 1 & -b & -c & bc \\ 1 & -c & -a & ca \\ 1 & -a & -b & ab \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

are equal. If, instead of evaluating each of the determinants, we try to transform one into the other by elementary transformations, we are led to the following generalization of this simple fact.

THEOREM 1. *If $a_1, \cdots, a_n$ are arbitrary complex numbers and $p_{ki} = (-1)^k \sum^{(i)} a_1 \cdots a_k$, the sum to be extended over all products k at a time of $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n$, then*

* E. R. Heineman, *Generalized Vandermonde determinants*, Transactions of this Society, vol. 31 (1929), p. 464.

† Muir, *Theory of Determinants*, vol. 4, p. 151; see also Muir and Metzler, *A Treatise on the Theory of Determinants*, p. 344.

‡ See, for example, MacMahon, *Combinatory Analysis*, vol. 1, p. 3.

§ For example, as in MacMahon, loc. cit., p. 4.

$$(1)\qquad \begin{vmatrix} 1 & p_{11} & \cdots & p_{n-1,1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 1 & p_{1n} & \cdots & p_{n-1,n} \end{vmatrix} = \begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 1 & a_n & \cdots & a_n^{n-1} \end{vmatrix}.$$

PROOF. The coefficients p_k of the equation of degree n, $x^n+p_1x^{n-1}+\cdots+p_n=0$, which has as roots the given numbers $a_1, a_2, \cdots, a_n$, are related to the numbers p_{ki} by the equation $p_k=p_{k,i}+a_ip_{k-1,i}$. If the equations obtained by writing this relation successively for k, $k-1, \cdots, 1$, be multiplied by $1, a_i, \cdots, a_i^{k-1}$, respectively, and the results are added, we obtain the formula

$$(2)\qquad p_{ki} = \sum_{j=0}^{k} a_i^j p_{k-j};$$

here it is to be understood that $p_0=p_{0i}=1$. This formula as derived is valid only for $0 \leqq k < n$. But it is immediately evident that it holds also for $k \geqq n$, *provided we put* $p_{ki}=0$, *for* $k \geqq n$. This will be our agreement throughout; it is in accord with the customary convention $p_k=0$, for $k>n$. Equation (1) is an immediate consequence of (2); for if we add to the $(k+1)$th column of the Vandermonde determinant, on the right side of (1), columns $1, 2, \cdots, k-1, k$ multiplied, respectively, by $p_k, p_{k-1}, \cdots, p_2, p_1$, it reduces to the $(k+1)$th column of the determinant on the left.

3. *Transformation of the Determinants in* (1). For our further purpose it is important to inquire how the left side of (1) can be transformed into the right side; the answer is given by solving equations (2) for a_i^j in terms of p_{ki}. This can be conveniently done in the following indirect way.

(a) If h_j represents the homogeneous product sum of degree j (that is the sum of all symmetric polynomials of degree j), then

$$(3)\qquad \sum_{j=0}^{k} h_j p_{k-j} = 0,$$

for $k \geqq 0$. For, from the identity

$$\prod_{i=1}^{n}(1 - a_ix) = \sum_{i=0}^{n} p_ix^i,$$

since for $|x| \leqq |1/a_i|$ we have

$$\frac{1}{1 - a_i x} = \sum_{j=0}^{\infty} a_i^j x^j,$$

and hence

$$\frac{1}{\prod_{i=1}^{n} (1 - a_i x)} = \prod_{i=1}^{n} \sum_{j=0}^{\infty} a_i^j x^j = \sum_{j=0}^{\infty} h_j x^j,$$

it follows that, for $|x| \leqq |1/a_i|$,

$$\sum_{i=0}^{n} p_i x^i \sum_{j=0}^{\infty} h_j x^j \equiv 1, \qquad (i = 1, \cdots, n).$$

Comparison of the coefficients of x^k shows that (3) is valid for any $k > 0$, since $p_k = 0$ for $k > n$; moreover $h_0 = 1$.

(b) If we write (2) in the form

$$a_i^k = p_{ki} - \sum_{j=0}^{k-1} a_i^j p_{k-j}, \tag{4}$$

it is easily proved that

$$a_i^k = \sum_{j=0}^{k} h_{k-j} p_{ji}, \qquad (0 \leqq k). \tag{5}$$

For $k = 0$, this reduces to the identity $1 \equiv 1$; for $k = 1$, to $a_i = h_1 + p_{1i}$, which is immediately verifiable. Assuming now that (5) holds for $k = 0, 1, \cdots, m$, we find by use of (4) and (3) that

$$\begin{aligned} a_i^{m+1} &= p_{m+1,i} - \sum_{j=0}^{m} a_i^j p_{m+1-j} = p_{m+1,i} - \sum_{j=0}^{m} \sum_{l=0}^{j} h_{j-l} p_{li} p_{m+1-j} \\ &= p_{m+1,i} - \sum_{l=0}^{m} \sum_{j=l}^{m} h_{j-l} p_{li} p_{m+1-j} \\ &= p_{m+1,i} - \sum_{l=0}^{m} p_{li} \sum_{j=0}^{m-l} h_j p_{m+1-j-l} \\ &= p_{m+1,i} + \sum_{l=0}^{m} p_{li} h_{m+1-l} = \sum_{l=0}^{m+1} h_{m+1-l} p_{li}. \end{aligned}$$

This completes the proof of (5); it shows that by using the multipliers $h_0, h_1, \cdots, h_{n-1}$ the determinant on the right of (1) is obtainable by elementary transformations from the one on the left.

4. *Transformation into Vandermonde Type.* But (5) can also be used to transform the determinant on the left of (1) into a generalized Vandermonde determinant. We shall designate the determinant on the left of (1) by its first row, thus $|1, p_{11}, \cdots, p_{n-1,1}|$. And we shall use the symbol $(0, m_1, m_2, \cdots, m_{n-1})$ to designate the generalized Vandermonde determinant whose ith row consists of the elements $1, a_i^{m_1}, a_i^{m_2}, \cdots, a_i^{m_{n-1}}$; we shall suppose $m_1 < m_2 < \cdots < m_{n-1}$. In this notation, Theorem 1 takes the form $|1, p_{11}, \cdots, p_{n-1,1}| = (0, 1, 2, \cdots, n-1)$. But we see furthermore that

$$(6) \quad h_{m_{n-1}-n+1} |1, p_{11}, \cdots, p_{n-1,1}| = (0, 1, 2, \cdots, n-2, m_{n-1})$$

for arbitrary m_{n-1}.* Since $\sum_{j=0}^{m_{n-1}} p_{ji} h_{m_{n-1}-j} = a_i^{m_{n-1}}$, the multipliers $h_{m_{n-1}}, h_{m_{n-1}-1}, \cdots, h_{m_{n-1}-n+1}$ applied to the successive columns on the left transform the last column into the column of m_{n-1}th powers; for (5) holds with the understanding that $p_{ji} = 0$ for $j \geqq n$. Repeated application of (5) completes the transformation.

As an intermediate step in this transformation we have therefore the equality†

$$(7) \quad h_{m_{n-1}-n+1} |1, p_{11}, \cdots, p_{n-1,1}| = |1, p_{11}, \cdots, p_{n-2,1} a_1^{m_{n-1}}|.$$

We seek now to determine multipliers $\alpha, \alpha_0, \alpha_1, \cdots, \alpha_{n-2}$ such that

$$\alpha a_i^{m_{n-1}} + \alpha_0 + \alpha_1 p_{1i} + \cdots + \alpha_{n-2} p_{n-2,i} = a_i^{m_{n-2}}.$$

By (5), this condition is equivalent to

$$\alpha \sum_{j=0}^{m_{n-1}} h_{m_{n-1}-j} p_{ji} + \sum_{j=0}^{n-2} \alpha_j p_{ji} - \sum_{j=0}^{m_{n-2}} h_{m_{n-2}-j} p_{ji} = 0,$$

or since $p_{ji} = 0$ for $j > n-1$, to

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* It is to be understood that $h_j = 0$, for $j < 0$; (6) is obviously true for $m_{n-1} < n-1$.

† This formula holds also if $h_{m_{n-1}-n+1} = 0$, although it can then not be obtained as above, because in that case $a_i^{m_{n-1}}$ is a linear combination of $1, p_{1i}, \cdots, p_{n-2,i}$.

$$(\alpha h_{m_{n-1}-n+1} - h_{m_{n-2}-n+1})p_{n-1,i} + \sum_{j=0}^{n-2}(\alpha h_{m_{n-1}-j} + \alpha_j - h_{m_{n-2}-j})p_{ji} = 0.$$

These multipliers are therefore uniquely determined by the equations

$$\alpha h_{m_{n-1}-n+1} - h_{m_{n-2}-n+1} = 0, \quad \alpha h_{m_{n-1}-j} + \alpha_j - h_{m_{n-2}-j} = 0,$$

for $j=0, \cdots, n-2$, if $h_{m_{n-1}-n+1} \neq 0$. The only one of them in whose actual value we are interested is α_{n-2}; for it we find

$$\alpha_{n-2} = \frac{h_{m_{n-1}-n+1}h_{m_{n-2}-n+2} - h_{m_{n-1}-n+2}h_{m_{n-2}-n+1}}{h_{m_{n-1}-n+1}}.$$

We find therefore from (7)

$$\begin{vmatrix} h_{m_{n-1}-n+1} & h_{m_{n-2}-n+1} \\ h_{m_{n-1}-n+2} & h_{m_{n-2}-n+2} \end{vmatrix} \cdot |1, p_{11}, \cdots, p_{n-1,1}| = |1, p_{11}, \cdots, p_{n-2,1}, a_1^{m_{n-2}}, a_1^{m_{n-1}}|, \tag{8}$$

and hence by repeated application of (5):

$$\begin{vmatrix} h_{m_{n-1}-n+1} & h_{m_{n-2}-n+1} \\ h_{m_{n-1}-n+2} & h_{m_{n-2}-n+2} \end{vmatrix} \cdot |1, p_{11}, \cdots, p_{n-1,1}| = (0, 1, 2, \cdots, n-3, m_{n-2}, m_{n-1}). \tag{9}$$

It is readily seen, as in the last footnote above, that (8), and hence (9), also holds if $h_{m_{n-1}-n+1}=0$, and if $h_{m_{n-1}-n+1}h_{m_{n-2}-n+2} - h_{m_{n-1}-n+2}h_{m_{n-2}-n+1}=0$. A straightforward induction now enables us to prove the general formula foreshadowed in (6) and (9). Nothing is involved except the determination of multipliers and the use of (5). Hence we shall merely state the result.

THEOREM 2. *The general Vandermonde determinant* $(0, 1, 2, \cdots, k, m_{k+1}, m_{k+2}, \cdots, m_{n-1})$ *for* $0 \leqq k \leqq n-1$ *is equal to the determinant* $|1, p_{11}, \cdots, p_{n-1,1}|$ *multiplied by the determinant*

$$\begin{vmatrix} h_{m_{n-1}-n+1}, & h_{m_{n-2}-n+1}, & \cdots, & h_{m_{k+1}-n+1} \\ \cdots & \cdots & \cdots & \cdots \\ h_{m_{n-1}-k-1}, & h_{m_{n-2}-k-1}, & \cdots, & h_{m_{k+1}-k-1} \end{vmatrix},$$

where h_j *represents the homogeneous product sum of degree* j.*

* A proof of a theorem closely akin to this theorem and of several special cases of it, by methods quite different from those used above, is found in Muir and Metzler, loc. cit., pp. 329–336.

While $m_{k+1}, \cdots, m_{n-1}$ are arbitrary positive integers, there is obviously only interest in the case in which they are distinct positive integers, and $m_{k+1} > k+1$; in all other cases the corresponding determinant vanishes.

5. *Application of Muir's Theorem.* We bring in now the theorem of Muir referred to in the introduction, according to which the product of an arbitrary symmetric polynomial in n variables by the Vandermonde determinant $(0, 1, 2, \cdots, n-1)$ can be expressed as a sum of generalized Vandermonde determinants of the type here considered. In combination with Theorems 1 and 2, we obtain then the following result.

THEOREM 3. *The arbitrary symmetric polynomial in n variables* $\sum a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}$, *where* α_i *are positive integers or zero, is equal to the sum of all determinants of the form*

$$\begin{vmatrix} h_{m_1-n+1}, & h_{m_2-n+1}, & \cdots, & h_{m_{n-1}-n+1} \\ \cdots & \cdots & \cdots & \cdots \\ h_{m_1-1}, & h_{m_2-1}, & \cdots, & h_{m_{n-1}-1} \end{vmatrix},$$

for which the sets of indices $m_1, m_2, \cdots, m_{n-1}$ *are obtained from the distinct permutations of the exponents* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *by increasing their successive elements by* $0, 1, 2, \cdots, n-1$ *and then dropping the first element provided it is* 0. *If this first element is different from* 0, *the set* $m_1, m_2, \cdots, m_{n-1}$ *is obtained by permuting the modified exponents until the least among them,* r, *is in first place, then diminishing all of them by* r *and dropping the first one; in this case the corresponding determinant is to be multiplied by* $(-1)^\rho (a_1, a_2, \cdots, a_n)^r$, *where* ρ *is the number of transpositions required to bring the least exponent in first place.*

PROOF. By Muir's theorem, the product of $\sum a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}$ by $(0, 1, 2, \cdots, n-1)$ is equal to the sum of all determinants of the form $(\alpha_1^{(i)}, \alpha_2^{(i)}+1, \cdots, \alpha_n^{(i)}+n-1)$, where the numbers $\alpha_1^{(i)}, \alpha_2^{(i)}, \cdots, \alpha_n^{(i)}$ range over the distinct permutations of $\alpha_1, \alpha_2, \cdots, \alpha_n$. If r is the least of all of the numbers $\alpha_1^{(i)}, \cdots, \alpha_n^{(i)}+n-1$ and if ρ transpositions bring it in first place, then $(\alpha_1^{(i)}, \cdots, \alpha_n^{(i)}+n-1) = (-1)^\rho (a_1, a_2, \cdots, a_n)^r \cdot (0, m_1, \cdots, m_{n-1})$, where $r, m_1+r, \cdots, m_{n-1}+r$ is a permutation of $\alpha_1^{(i)}, \cdots, \alpha_n^{(i)}+n-1$. Theorem 3 follows now immediately from Theorems 1 and 2.

To express the homogeneous product sums in terms of elementary symmetric functions one can use the formula given by MacMahon and referred to in the introduction, the recursion formula (3), or the following explicit formula readily derivable from (3):

$$h_j = -\begin{vmatrix} 1, & 0, & \cdots, 0, & p_1 \\ p_1, & 1, & \cdots, 0, & p_2 \\ p_2, & p_1, & \cdots, 0, & p_3 \\ \cdots & \cdots & \cdots & \cdots \\ p_{j-1}, & p_{j-2}, & \cdots, p_1, & p_j \end{vmatrix}.$$

6. *Example.* To determine $\sum a^4b^3$ in 5 variables, that is, $\sum a^4b^3c^0d^0e^0$, we have $\alpha_1=4$, $\alpha_2=3$, $\alpha_3=\alpha_4=\alpha_5=0$. Hence there are 20 distinct permutations. Of these, the ones which contain any of the sequences 4 3, 4 x x x 0, or 3 x x 0 give rise to sets in which at least two elements are equal. There are left 9 permutations which lead to the following sets when their elements are increased by 0 1 2 3 4: (4 1 2 3 7), (0 5 2 6 4), (0 5 2 3 7), (0 1 6 3 7), (3 1 2 7 4), (0 1 5 7 4), (0 4 2 3 8), (0 1 5 3 8), and (0 1 2 6 8). Of these sets the first and fifth give rise to the same result, namely, to $-a_1\,a_2\,a_3\,a_4\,a_5\,h_2$; the remaining sets each lead to a 4th-order determinant. Development of these determinants gives

$$\sum a^4b^3c^0d^0e^0 = 2p_5h_2 - h_1^3h_4 + 3h_1^2h_2h_3 + 3h_1^2h_5 - h_1h_2^3 - 2h_1h_2h_4 - 5h_1h_3^2 - h_1h_6 + h_2^2h_3 - 3h_2h_5 + 7h_3h_4 - h_7.$$

Reduction to the p's leads, since $p_6=p_7=0$, to the result

$$\sum a^4b^3c^0d^0e^0 = -3p_1^3p_4 + 3p_1^2p_2p_3 + 7p_1^2p_5 - p_1p_2^3 + 2p_1p_2p_4 - 5p_1p_3^2 + p_2^2p_3 - 7p_2p_5 + 5p_3p_4.$$

SWARTHMORE COLLEGE

BLOCH'S THEOREM FOR MINIMAL SURFACES*

BY E. F. BECKENBACH†

The following theorem was first proved by Bloch.‡

BLOCH'S THEOREM. *There exists a positive absolute constant B with the following property. Let $Z=f(z)$ be analytic for $|z|\leqq 1$, with $|f'(0)|\geqq 1$; then in the Z-plane there is an open circle of radius at least B, which is the uniplanar§ map of a portion of the circle $|z|<1$.*

Other proofs of greater simplicity have been given. The present generalization follows the proof given by Landau and by Valiron.‖ In this paper we shall prove the following theorem.

THEOREM. *There exists a positive absolute constant B with the following property. Let the circle $u^2+v^2\leqq 1$ be mapped conformally on a minimal surface, with $\mathcal{E}_0\geqq 1$, where $\mathcal{E}_0$ denotes the area deformation ratio at the origin; then on the minimal surface there is an open geodesic circle of radius at least B, containing no singular points, which is the one-to-one map of a portion of the circle $u^2+v^2<1$.*

That is, there is a point on the surface such that no curve on the surface, issuing from this point and of length less than B, comes either to the boundary of the map or to a point where the conformal character of the map breaks down.

In order that the real analytic functions

$$x_j = x_j(u, v), \qquad (j = 1, 2, 3),$$

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* Presented to the Society, February 25, 1933.

† National Research Fellow.

‡ *Les théorèmes de M. Valiron sur les fonctions entières, et la théorie de l'uniformisation*, Comptes Rendus, vol. 178 (1924), pp. 2051–2052, and Annales de la Faculté des Sciences de l'Université de Toulouse, (3), vol. 17 (1925), pp. 1–22.

§ German *schlicht*.

‖ Landau, *Über die Blochsche Konstante und zwei verwandte Weltkonstanten*, Mathematische Zeitschrift, vol. 30 (1929), pp. 608–634; Valiron, *Sur le théorème de M. Bloch*, Rendiconti del Circolo Matematico di Palermo, vol. 54 (1930), pp. 76–82.

shall give a conformal map of the domain of definition, the parameters must be isothermic; that is,

$$E = G, \qquad F = 0. \tag{1}$$

With this choice of parameters, by a theorem of Weierstrass, a necessary and sufficient condition that the surface be minimal is that the x_j be harmonic. This being so, these functions are the real parts of analytic functions of a complex variable,

$$x_j = \Re f_j(z), \qquad z = u + iv. \tag{2}$$

Equations (1) now are equivalent to

$$\sum_{j=1}^{3} f_j'^2 = 0. \tag{3}$$

We shall let these functions (2) represent the mapping in the above theorem. It is no restriction on the generality that we take $f_j(0) = 0$, so that

$$f_j(z) = \sum_{t=1}^{\infty} a_{j,t} z^t.$$

The minimal surface defined by the equation $y_j = \Im f_j(z)$, where $\Im f_j(z)$ designates the imaginary part of $f_j(z)$, is called the adjoint of the minimal surface defined by (2). The surface and its adjoint are applicable to each other, the element of length on each being given by

$$E^{1/2} \left| dz \right| \equiv \left[\tfrac{1}{2} \sum_{j=1}^{3} \left| f_j' \right|^2 \right]^{1/2} \left| dz \right| .$$

LEMMA 1. *Let the circle $|w| < R$ be mapped conformally on a minimal surface, with $\mathcal{E}_0 > 0$, where $\mathcal{E}_0$ designates the area deformation ratio at the origin. Then it is possible to choose rectangular axes, with origin at the image of $w = 0$, so that in the coordinate functions of the surface,*

$$x_j = \Re g_j(w) = \Re \sum_{t=1}^{\infty} b_{j,t} w^t, \qquad w = p + iq,$$

we have

$$\left| b_{1,1} \right| = \left| b_{2,1} \right| = \left| b_{3,1} \right| = (\tfrac{2}{3}\mathcal{E}_0)^{1/2} = a > 0. \tag{4}$$

Since $\mathcal{E}_0 > 0$, the surface has a definite tangent plane and normal at the image of $w=0$. We choose coordinate axes, with origin at the image of $w=0$, in such a way that the normal at this point has equal components on the three coordinate axes. For these axes, let the coordinate functions of the surface be given by

$$x_j = \mathcal{R} g_j(w) = \mathcal{R} \sum_{t=1}^{\infty} b_{j,t} w^t, \qquad |w| < R;$$

then, as in (3),

$$\sum_{j=1}^{3} g_j'^2 = 0. \tag{5}$$

Let $X_j(w)$, $(j=1, 2, 3)$, denote the direction cosines of the normal to the surface at the point corresponding to w. Because of our choice of axes, we have

$$X_1(0) = X_2(0) = X_3(0) \neq 0. \tag{6}$$

Now

$$g_j' = \frac{\partial x_j}{\partial p} - i \frac{\partial x_j}{\partial q};$$

multiplying this equation through by X_j and summing with respect to j, we get

$$\sum_{j=1}^{3} X_j g_j' = 0. \tag{7}$$

Evaluating (7) at $w=0$, and using (6), we obtain

$$\sum_{j=1}^{3} g_j'(0) = 0. \tag{8}$$

We have

$$\left(\sum_{j=1}^{3} g_j' \right)^2 = \sum_{j=1}^{3} g_j'^2 + 2 \sum_{j<k}^{3} g_j' g_k',$$

so that, by (5) and (8),

$$\sum_{j \leq k}^{3} g_j'(0) g_k'(0) = 0. \tag{9}$$

Let $\alpha^3+m_1\alpha^2+m_2\alpha+m_3=0$ be the equation whose roots are $g_1'(0)$, $g_2'(0)$, $g_3'(0)$. Then, by (8) and (9), $m_1=m_2=0$, so that $\alpha^3+m_3=0$. That is, $g_1'(0)$, $g_2'(0)$, $g_3'(0)$ lie on a circle with center at the origin, or, what is the same thing, $|b_{1,1}|=|b_{2,1}|=|b_{3,1}|$. From

$$\mathcal{E}=\tfrac{1}{2}\sum_{j=1}^{3}|g_j'|^2,$$

we therefore obtain (4).

LEMMA 2. *Let the functions*

$$g_j(w)=\sum_{t=1}^{\infty}b_{j,t}w^t, \qquad (j=1,2,3), \tag{10}$$

be analytic for $|w|<R$, *with*

$$\sum_{j=1}^{3}g_j'^2=0, \qquad \tfrac{1}{2}\sum_{j=1}^{3}|b_{j,1}|^2=\mathcal{E}_0>0,$$

and

$$|g_j(w)|\leqq 2^{1/2}M. \tag{11}$$

Then on the minimal surface defined by $x_j=\Re g_j(w)$ *there is an open geodesic circle whose center is the image of the origin, whose radius is at least* $R^2\mathcal{E}_0/(6\cdot 3^{1/2}\cdot M)$, *and which contains no singular points. Similarly, there is a geodesic circle of the same description on the adjoint minimal surface.*

The condition (11) necessitates that both the surface and its adjoint be contained in a sphere with center at the origin and radius $6^{1/2}M$; conversely, (11) surely is satisfied if both surfaces are contained in a sphere with center at the origin and radius M.

By simply rotating the coordinate axes, we can normalize the functions (10) in accordance with Lemma 1, without altering the conditions of Lemma 2. We consider this done and still use the expressions (10), so that (4) holds for the functions (10).

Landau has shown* that $g_j(w)$ gives a uniplanar map of

$$|w|\leqq\frac{aR}{4\cdot 2^{1/2}\cdot M}=b, \tag{12}$$

and that this map contains the uniplanar open circle

* Loc. cit., pp. 616–617.

$$|g_j| < \frac{a^2R^2}{6\cdot 2^{1/2}\cdot M},$$

the center of this circle being the image of the origin.

The length of a curve on the minimal surface (10) or on its adjoint is given by the equation

$$L = \int \left[\tfrac{1}{2}\sum_{j=1}^{3} |g_j'|^2\right]^{1/2} |dw| .$$

By Minkowski's inequality,*

$$L^2 \geqq \tfrac{1}{2}\sum_{j=1}^{3}\left[\int |g_j'|\cdot|dw|\right]^2,$$

the path of integration being the same throughout. We have just seen that, for all paths of integration,

$$\text{minimum}\int_0^b |g_j'|\cdot|dw| \geqq \frac{a^2R^2}{6\cdot 2^{1/2}\cdot M},$$

so that, a fortiori,

$$\text{minimum}\int_0^b \left[\tfrac{1}{2}\sum_{j=1}^{3}|g_j'|^2\right]^{1/2}|dw| \geqq \frac{3^{1/2}\cdot a^2R^2}{2\cdot 6\cdot M} = \frac{R^2\mathcal{E}_0}{6\cdot 3^{1/2}\cdot M}.$$

This demonstrates the existence of the prescribed geodesic circle. Singular points on the minimal surface occur only where $\mathcal{E}=0$; that is, where simultaneously $g_1'=g_2'=g_3'=0$. But $g_j(w)$ gives a uniplanar map of (12) and therefore $g_j'\neq 0$ in this region. It follows that $\mathcal{E}\neq 0$ in (12).

LEMMA 3.† *Let ξ be a constant, $|\xi|<1$, and let*

$$z = \frac{\xi + w}{1 + \bar{\xi}w}.$$

Then, for $|w|<1$,

$$\left|\frac{dz}{dw}\right| = \frac{1-|z|^2}{1-|w|^2}.$$

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* See, for instance, Pólya und Szegö, *Aufgaben und Lehrsätze*, vol. I, 1925, p. 56, §91.

† For the elementary proof, see for instance Landau, loc. cit., p. 617.

We come now to the proof of the theorem. We can assume that $E_0=1$, because if the theorem holds in this case it holds a fortiori for $E_0>1$.

Let

$$N = \text{maximum}_{|z|\leqq 1}(1-|z|^2)\left[\tfrac{1}{2}\sum_{j=1}^{3}|f_j'|^2\right]^{1/2}$$

$$= \text{maximum}_{|z|\leqq 1}(1-|z|^2)E^{1/2}.$$

Then $N\geqq 1$, and N is attained for some $z=\xi$, with $|\xi|<1$. The unit circle is mapped conformally on itself by

$$z=\frac{\xi+w}{1+\bar{\xi}w},$$

so that the functions

$$g_j(w)=\frac{f_j(z)}{N}$$

are analytic for $|w|\leqq 1$. And the real parts of these functions map $|w|\leqq 1$ conformally on a minimal surface. By Lemma 3, for $|w|<1$, we have

$$(1-|w|^2)|g_j'(w)| = (1-|w|^2)\frac{1}{N}|f_j'(z)|\cdot\left|\frac{dz}{dw}\right|$$

$$=\frac{1}{N}(1-|z|^2)|f_j'(z)|,$$

whence

$$(13)\qquad (1-|w|^2)\left[\tfrac{1}{2}\sum_{j=1}^{3}|g_j'(w)|^2\right]^{1/2}$$

$$=\frac{1}{N}(1-|z|^2)\left[\tfrac{1}{2}\sum_{j=1}^{3}|f_j'(z)|^2\right]^{1/2}\leqq 1,$$

so that

$$\left[\tfrac{1}{2}\sum_{j=1}^{3}|g_j'(0)|^2\right]^{1/2}=\mathcal{E}_0=\frac{1}{N}(1-|\xi|^2)\left[\tfrac{1}{2}\sum_{j=1}^{3}|f_j'(\xi)|^2\right]^{1/2}=1.$$

From (13) we obtain also, for $|w| \leqq 1/2$,

$$\left[\tfrac{1}{2}\sum_{j=1}^{3}|g_j'(w)|^2\right]^{1/2} \leqq \frac{4}{3},$$

so that, integrating along any radius, we have

$$L = \int_0^{1/2}\left[\tfrac{1}{2}\sum_{j=1}^{3}|g_j'(w)|^2\right]^{1/2} dr \leqq \frac{2}{3}.$$

Therefore the minimal surfaces given by $x_j = \Re g_j(w)$ and $y_j = \Im g_j(w)$ for $|w| \leqq 1/2$ both are contained in a sphere with center at the origin and radius $2/3$.

Consequently, we can apply Lemma 2 to the functions $g_j(w)$, with $R = 1/2$, $M = 2/3$, $\mathcal{E}_0 = 1$. We see, namely, that on the map of $|w| < 1/2$ given by

$$x_j = \Re g_j(w), \qquad (j = 1, 2, 3),$$

there is an open geodesic circle of radius at least $1/(16 \cdot 3^{1/2})$ and comprising no singular points; and therefore that on the map of $|z| < 1$ given by (2) there is an open geodesic circle of radius at least

$$\frac{N}{16 \cdot 3^{1/2}} \geqq \frac{1}{16 \cdot 3^{1/2}}$$

and comprising no singular points.

THE OHIO STATE UNIVERSITY

CONVERGENCE FACTORS FOR DOUBLE SERIES*

BY W. H. DURFEE

1. *Introduction.* By a theorem due originally to Frobenius† if the power series $y(z)=\sum_{i=0}^{\infty}a_iz^i$ has the unit circle as circle of convergence, and if $\sum_{i=0}^{\infty}a_i$ is summable by Cesàro's first mean with the value s, then $\lim y(z)=s$ as $z\to+1$ along any path lying between two fixed chords intersecting at $z=+1$. This theorem has been considerably extended, in the field of double series notably by Bromwich and Hardy,‡ and by C. N. Moore.§ The former proved that if $f(x, y)=\sum_{i,j=0}^{\infty}a_{ij}x^iy^j$, and if $|S_{ij}^{(k)}|$, the kth Hölder mean of $\sum a_{ij}$, is bounded for all values of i and j, and $\lim_{i,j\to\infty}S_{ij}^{(k)}=s$, then also $\lim_{x,y\to 1}f(x, y)=s$. More particular reference will presently be made to Moore's paper, his theorems being the starting point for the present article. Robison,‖ also, has given necessary and sufficient conditions for the regularity of a transformation applied to a double sequence.

The writer, in a paper on series of the form $y(z)=\sum_{i=0}^{\infty}a_iz^{f(i)}$, gave sufficient conditions on $f(i)$ so that $\lim_{z\to 1}y(z)=s$.¶ The present paper deals with double series of the type

$$J(z, w)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}z^{f(i)}w^{g(j)},$$

where z, w are complex variables, and $f(i)$, $g(j)$ are logarithmico-exponential functions,** called for brevity L-functions. Sufficient conditions on $f(i)$, $g(j)$ will be given so that if $\sum a_{ij}$ is summable $(C, r-1)$ with the value s, then $J(z, w)$ will be convergent for $|z|<1$, $|w|<1$, and $\lim_{(z,w)\to(1,1)}J(z, w)=s$.

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* Presented to the Society, April 8, 1932.

† Journal für Mathematik, vol. 89 (1880), p. 262.

‡ Proceedings of the London Mathematical Society, (2), vol. 2 (1904), p. 161.

§ Transactions of this Society, vol. 29 (1927), p. 227.

‖ Transactions of this Society, vol. 28 (1926), p. 50.

¶ American Journal of Mathematics, vol. 53 (1931), p. 817.

** Hardy, *Orders of Infinity.*

2. *Notation.* We shall employ Moore's notation. Thus

$$(1)\quad s_{m_1m_2} = \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} a_{ij},$$

$$(2)\quad S^{(k)}_{m_1m_2} = \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} \frac{\Gamma(k+m_1-i)}{\Gamma(k)\cdot\Gamma(m_1-i+1)}\cdot\frac{\Gamma(k+m_2-j)}{\Gamma(k)\cdot\Gamma(m_2-j+1)}s_{ij},$$

$$(3)\quad A^{(k)}_{m_1m_2} = \frac{\Gamma(m_1+k)}{\Gamma(k+1)\cdot\Gamma(m_1)}\cdot\frac{\Gamma(m_2+k)}{\Gamma(k+1)\cdot\Gamma(m_2)}.$$

If the quotient $S^{(k)}_{m_1m_2}/A^{(k)}_{m_1m_2}$ approaches a limit s as m_1, m_2 become infinite, we say that the series $\sum a_{ij}$ is summable (C, k) with the value s. We shall also have occasion to employ the following notation:

$$(4)\quad \phi_{ij}(z, w) = z^{f(i)}w^{g(j)},$$

$$(5)\quad \phi^{(p,q)}_{ij}(z, w) = \frac{\partial^{p+q}\phi_{ij}(z, w)}{\partial i^p\partial j^q},$$

$$(6)\quad \Delta_{rr}\phi_{ij}(z, w) = \sum_{s_1=0}^{r}\sum_{s_2=0}^{r}(-1)^{s_1}(-1)^{s_2}\binom{r}{s_1}\binom{r}{s_2}\phi_{i+s_1,j+s_2}(z,w),$$

$$(7)\quad \Delta_{r0}\phi_{ij}(z, w) = \sum_{s_1=0}^{r}(-1)^{s_1}\binom{r}{s_1}\phi_{i+s_1,j}(z, w).$$

The region within which $|z|<1$, $|w|<1$, will be denoted by $E(z, w)$, and the open region in the neighborhood of $(1, 1)$ lying between two chords of the unit circle intersecting at $+1$, by $E'(z, w)$.

THEOREM. *If $\sum a_{ij}$ is summable $(C, r-1)$ with the value s, when $r\geqq 1$ is an integer, and if*

(a) $|S^{(r-1)}_{ij}/A^{(r-1)}_{ij}|<C$, $\quad (i, j=1, 2, \cdots; C \text{ a constant})$;

(b) *$f(t)$, $g(t)$ are L-functions which, together with their first $(r-1)$ derivatives, exist and are continuous, are of constant sign, and are monotonic for $t\geqq 1$;*

(c) $\quad \log t = \sigma[f(t)]$, $\log t = \sigma[g(t)]$;

(d) $\quad f(t) = \sigma(t^\alpha)$, $g(t) = \sigma(t^\alpha)$ *for some* $\alpha > 0$;

then the double series $J(z, w)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}z^{f(i)}w^{g(j)}$ will converge in $E(z, w)$, and $\lim_{(z,w)\to(1,1)} J(z, w)=s$, the paths of approach lying in $E'(z, w)$.

3. *Statement of Lemmas.* For the sake of brevity the following lemmas are stated here without proof.*

LEMMA 1. *As* $(z, w)\to(1, 1)$ *in* $E'(z, w)$, $|\log z| = O[\log \rho]$, $|\log w| = O(\log \tau)$, *where* $\rho = |z|$, $\tau = |w|$.

LEMMA 2. *If* $h(t)$ *satisfies conditions* (c) *and* (d) *of the Theorem, then for* $k \geqq 1$, $h^{(k)}(t)/h(t) = O(1/t^k)$, *and* $h^{(k)}(t)/h'(t) = O(1/t^{k-1})$.

4. *General Relations.* Each term $T[\phi_{ij}^{(p,q)}(\rho,\tau)]$ of $\phi_{ij}^{(p,q)}(\rho,\tau)$ is of the form

$$(8)\qquad B_1\rho^{f(i)}\tau^{g(j)}\prod_{\lambda=1}^{p}[f^{(\lambda)}(i)]^{\alpha_\lambda}\prod_{\sigma=1}^{q}[g^{(\sigma)}(j)]^{\beta_\sigma}(\log\rho)^{\alpha}(\log\tau)^{\beta},$$

where B_1 is a constant, $\alpha=\sum_{\lambda=1}^{p}\alpha_\lambda$, $p=\sum_{\lambda=1}^{p}\lambda\alpha_\lambda$, $\beta=\sum_{\sigma=1}^{q}\beta_\sigma$, $q=\sum_{\sigma=1}^{q}\sigma\beta_\sigma$, and any, or all but one, of α_λ or β_σ may be zero. It will be noted that $p\geqq\alpha$, $q\geqq\beta$. By Lemma 2 we have

$$(9)\qquad |T[\phi_{ij}^{(p,q)}(\rho,\tau)]| \leqq B_2\rho^{f(i)}\tau^{g(j)}\frac{[f(i)]^{\alpha}[g(j)]^{\beta}}{i^p j^q}|\log\rho|^{\alpha}|\log\tau|^{\beta},$$

$$(10)\qquad |T[\phi_{ij}^{(p,q)}(\rho,\tau)]| \leqq B_3\rho^{f(i)}\tau^{g(j)}\frac{[f'(i)]^{\alpha}[g'(j)]^{\beta}}{i^{p-\alpha}j^{q-\beta}}|\log\rho|^{\alpha}|\log\tau|^{\beta}.$$

By Lemma 1, in $E'(z, w)$, $|\phi_{ij}^{(p,q)}(z, w)| \leqq B_4|\phi_{ij}^{(p,q)}(\rho,\tau)|$; so that if we denote by $\sum T_{ij}^{(p,q)}(\rho,\tau)$ the sum of all terms of $\phi_{ij}^{(p,q)}(\rho,\tau)$ whose signs are unlike that of $\rho^{f(i)}\tau^{g(j)}[f(i)\log\rho]^{p}[g(j)\log\tau]^{q}$, the leading term, we have

$$(11)\qquad \Phi \leqq (-1)^p(-1)^q B_4\{\phi_{ij}^{(p,q)}(\rho,\tau) - 2\sum T_{ij}^{(p,q)}(\rho,\tau)\},$$

where $\Phi = |\phi_{ij}^{(p,q)}(z, w)|$. From (6) and (11) we obtain

$$|\Delta_{rr}\phi_{ij}(z, w)| \leqq B_4\int_0^1 d\xi_1\int_0^1 d\eta_1\cdots$$

$$(12)\qquad \cdot\int_0^1 d\xi_r\int_0^1\{\phi_{\mu\nu}^{(r,r)}(\rho,\tau) - 2\sum T_{\mu\nu}^{(r,r)}(\rho,\tau)\}d\eta_r,$$

where $\mu = i+\xi_1+\cdots+\xi_r$, $\nu = j+\eta_1+\cdots+\eta_r$. By (9), since $i\leqq\mu$, $\rho^{f(u)}\leqq\rho^{f(i)}$, $f(\mu)\leqq f(i+r)$, with similar inequalities for j and ν, we have for fixed (z, w), if we set $M = |\Delta_{rr}\phi_{ij}(z, w)|$,

* The proof of Lemma 1 may be found in my paper cited above; Lemma 2 may be deduced from certain remarks by Hardy, in his *Orders of Infinity*.

$$M \leqq B_5 \int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots \int_0^1 d\xi_r \int_0^1 \rho^{f(\mu)} \tau^{g(\nu)} \frac{[f(\mu)]^\alpha [g(\nu)]^\beta}{\mu^r \nu^r} d\eta_r$$

$$\leqq B_5 \rho^{f(i)} \tau^{g(j)} \frac{[f(i+r)]^r [g(j+r)]^r}{i^r j^r}. \tag{13}$$

It can be shown easily that if a, b are positive constants

$$\lim_{i\to\infty} \rho^{f(i)} [f(i+a)]^b = \lim_{j\to\infty} \tau^{g(j)} [g(j+a)]^b = 0; \tag{14}$$

whence

$$\lim_{i,j\to\infty} |\Delta_{rr}\phi_{ij}(z, w)| = \lim_{i\to\infty} |\Delta_{rr}\phi_{ij}(z, w)| = \lim_{j\to\infty} |\Delta_{rr}\phi_{ij}(z, w)| = 0. \tag{15}$$

5. *Proof of Theorem.* C. N. Moore* has given necessary and sufficient conditions that a double series $\sum a_{ij}F_{ij}(z, w)$ shall converge in $E(z, w)$ and approach a limit s as $(z, w)\to(1, 1)$ in $E'(z, w)$, the series $\sum a_{ij}$ being summable $(C, r-1)$ with the value s, and condition (a) of the Theorem being satisfied. For series of our type, $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} z^{f(i)} w^{g(j)}$, these conditions are:

(A) $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^{r-1}j^{r-1}|\Delta_{rr}\phi_{ij}(z, w)| < K(z, w)$, $(E(z, w))$;

(B$_1$) $\lim_{j\to\infty} j^{r-1}\sum_{i=1}^{p} i^{r-1}|\Delta_{r0}\phi_{ij}(z, w)| = 0$, $(E(z, w); p = 1, 2, \cdots)$;

(B$_2$) $\lim_{i\to\infty} i^{r-1}\sum_{j=1}^{q} j^{r-1}|\Delta_{0r}\phi_{ij}(z, w)| = 0$, $(E(z, w); q = 1, 2, \cdots)$;

(C) $i^{r-1}j^{r-1}|\phi_{ij}(z, w)| < M(z, w)$, $(E(z, w); i, j = 1, 2, \cdots)$;

(A′) $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^{r-1}j^{r-1}|\Delta_{rr}\phi_{ij}(z, w)| < K$, $(E'(z, w))$;

(D$_1$) $\lim_{(z,w)\to(1,1)} \sum_{j=q}^{\infty} j^{r-1}|\Delta_{rr}\phi_{ij}(z, w)| = 0$, $(i, q = 1, 2, \cdots)$;

(D$_2$) $\lim_{(z,w)\to(1,1)} \sum_{i=p}^{\infty} i^{r-1}|\Delta_{rr}\phi_{ij}(z, w)| = 0$, $(p, j = 1, 2, \cdots)$;

(E) $\lim_{(z,w)\to(1,1)} \phi_{ij}(z, w) = 1$, $(i, j = 1, 2, \cdots)$;

* Loc. cit.

where $K(z, w)$ and $M(z, w)$ are finite for each (z, w) in $E(z, w)$, and K is a positive constant. We proceed to show that these eight conditions are fulfilled in the present case.

CONDITION (A). By (12), since $i \leqq \mu, j \leqq \nu$,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^{r-1}j^{r-1}\left|\Delta_{rr}\phi_{ij}(z, w)\right| \leqq B_4 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots$$

$$\cdot \int_0^1 d\xi_r \int_0^1 \mu^{r-1}\nu^{r-1}\left\{\phi_{\mu\nu}^{(r,r)}(\rho, \tau) - 2\sum T_{\mu\nu}^{(r,r)}(\rho, \tau)\right\} d\eta_r. \tag{16}$$

Considering first that part of this integrand involving $\phi_{\mu\nu}^{(r,r)}$, and integrating by parts with respect to η_r and then with respect to ξ_r, we obtain

$$B_4 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots \int_0^1 d\xi_r \int_0^1 \mu^{r-1}\nu^{r-1}\phi_{\mu\nu}^{(r,r)}(\rho, \tau) d\eta_r$$

$$= B_4 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left\{G(i, j) - G(i+1, j) - G(i, j+1) + G(i+1, j+1)\right\}, \tag{17}$$

where

$$G(i, j) = \int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots \int_0^1 d\xi_{r-1} \int_0^1 \sum_{s=0}^{r-1}\sum_{t=0}^{r-1}(-1)^s(-1)^t$$

$$\cdot \frac{(r-1)!}{(r-1-s)!}\frac{(r-1)!}{(r-1-t)!}\mu_0^{r-1-s}\nu_0^{r-1-t}\phi_{\mu_0\nu_0}^{(r-1-s,r-1-t)}(\rho, \tau) d\eta_{r-1}, \tag{18}$$

in which expression $\mu_0 = i + \xi_1 + \cdots + \xi_{r-1}$, and $\nu_0 = j + \eta_1 + \cdots + \eta_{r-1}$. By the aid of (9) we find

$$G(i, j) \leqq B_6[(r-1)!]^2 \int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots$$

$$\cdot \int_0^1 d\xi_{r-1} \int_0^1 \sum_{s=0}^{r-1}\sum_{t=0}^{r-1}\rho^{f(\mu_0)}\tau^{g(\nu_0)}\left[f(\mu_0)\right]^{r-1-s}\left[g(\nu_0)\right]^{r-1-t} d\eta_{r-1}$$

$$\leqq B_6(r!)^2\rho^{f(i)}\tau^{g(j)}\left[f(i+r-1)\cdot g(j+r-1)\right]^{r-1}. \tag{19}$$

This expression, by virtue of (14), approaches zero when i, or j, or both, increase indefinitely; so that

$$B_4 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left\{G(i, j) - G(i+1, j) - G(i, j+1) + G(i+1, j+1)\right\}$$

$$= B_4 G(1, 1) \leqq B_7(r!)^2\rho^{f(1)}\tau^{g(1)}\left[f(r)\cdot g(r)\right]^{r-1}. \tag{20}$$

Thus (17) is bounded for fixed z and w.

Returning now to the remaining part of the integrand in (16), we have, by (10), for each term

$$(21)\qquad \begin{aligned} &2B_4 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots \int_0^1 d\xi_r \int_0^1 - \mu^{r-1}\nu^{r-1} T_{\mu\nu}^{(r,r)}(\rho,\tau) d\eta_r \\ &\qquad \leqq 2B_4 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots \\ &\cdot \int_0^1 d\xi_r \int_0^1 B_2 \mu^{\alpha-1}\nu^{\beta-1}\rho^{f(\mu)}\tau^{g(\nu)}\left[f'(\mu)\right]^{\alpha}\left[g'(\nu)\right]^{\beta}\left|\log \rho\right|^{\alpha}\left|\log \tau\right|^{\beta} d\eta_r. \end{aligned}$$

This integrand is, except for a constant factor, the leading term of $\mu^{\alpha-1}\nu^{\beta-1}\phi_{\mu\nu}^{(\alpha,\beta)}(\rho,\tau)$. Now $T_{\mu\nu}^{(r,r)}(\rho,\tau)$, being negative, cannot be the leading term of $\phi_{\mu\nu}^{(r,r)}$; hence $\phi_{\mu\nu}^{(\alpha,\beta)}$ is of lower order than $\phi_{\mu\nu}^{(r,r)}$, and may be substituted for $T_{\mu\nu}^{(r,r)}$. We now set up a new expression, like (16) but with $\phi_{\mu\nu}^{(\alpha,\beta)}$ in place of $\phi_{\mu\nu}^{(r,r)}$, and sufficient repetition of this process must eventually lead, by (20), to

$$(22)\qquad \begin{aligned} &B_8 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots \\ &\qquad \cdot \int_0^1 d\xi_r \int_0^1 \phi_{r,\mu}^{(1,1)}(\rho,\tau) d\eta_r \leqq B_9 \rho^{f(1)}\tau^{g(1)}. \end{aligned}$$

We have, therefore,

$$(23)\qquad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^{r-1}j^{r-1}\left|\Delta_{rr}\phi_{ij}(z,w)\right| < B_{10}(r!)^2\rho^{f(1)}\tau^{g(1)}\left[f(r)\cdot g(r)\right]^{r-1},$$

which proves that condition (A) is satisfied. By an entirely similar procedure we find that

$$(24)\qquad \begin{aligned} &j^{r-1}\sum_{i=1}^{p} i^{r-1}\left|\Delta_{r0}\phi_{ij}(z,w)\right| \\ &\qquad \leqq B_{11}j^{r-1}\sum_{i=1}^{p}\int_0^1 d\xi_1 \int_0^1 d\xi_2 \cdots \int_0^1 d\xi_{r-1}\int_0^1 \mu^{r-1} \\ &\qquad \cdot \left\{\phi_{\mu j}^{(r,0)}(\rho,\tau) - 2\sum T_{\mu j}^{(r,0)}(\rho,\tau)\right\} d\xi_r \\ &\qquad < B_{12}j^{(r-1)}(r!)\tau^{g(j)}\left\{\rho^{f(p+1)}\left[f(p+r)\right]^{r-1} - \rho^{f(1)}\left[f(r)\right]^{r-1}\right\}. \end{aligned}$$

By (14) $\lim_{p\to\infty} \rho^{f(p+1)}[f(p+r)]^{r-1}=0$, so that the expression within the braces is bounded for $p\geqq 1$; and since, by condition (c) of the Theorem, $\lim_{j\to\infty} j^{r-1}\tau^{g(j)}=0$, we have

$$\lim_{j\to\infty} j^{r-1}\sum_{i=1}^{p} i^{r-1}\left|\Delta_{r0}\phi_{ij}(z, w)\right| = 0, \quad (E(z, w); p = 1, 2, \cdots).$$

Condition (B_1) is therefore satisfied. The argument for (B_2) is, of course, precisely similar.

Proceeding to condition (C), we note that by condition (c) of the Theorem, for an assigned $\epsilon>0$ there exist i_0, j_0, such that for $i>i_0$, $\log i<\epsilon f(i)$, and for $j>j_0$, $\log j<\epsilon g(j)$. By choosing ϵ less than both $|\log\rho|/(r-1)$, and $|\log\tau|/(r-1)$ we have, for such i and j, $\epsilon(r-1)f(i)<f(i)|\log\rho|$, $\epsilon(r-1)g(j)<g(j)|\log\tau|$, and hence

$$(25)\qquad \begin{aligned} i^{r-1}j^{r-1}\left|\phi_{ij}(z, w)\right| &= i^{r-1}j^{r-1}\rho^{f(i)}\tau^{g(j)} \\ &< e^{(r-1)\{\log i-\epsilon f(i)\}}\cdot e^{(r-1)\{\log j-\epsilon g(j)\}} < 1. \end{aligned}$$

Condition (C) is therefore satisfied.

In condition (A), the bound $K(z, w)$ depends upon z and w, for the constant B_{10} in (23) depends upon $\log\rho$ and $\log\tau$. We now further define $E'(z, w)$ as follows. For a given L, $0<L<1$, let all values of (z, w) in $E'(z, w)$ be such that $|\log\rho|\leqq|\log L|$, $|\log\tau|\leqq|\log L|$. If we now set B_{13} equal to the value of B_{10} corresponding to $\rho=\tau=L$, we have, for all (z, w) in $E'(z, w)$,

$$(26)\qquad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^{r-1}j^{r-1}\left|\Delta_{rr}\phi_{ij}(z, w)\right| < B_{13}(r!)^2[f(r)\cdot g(r)]^{r-1} = K.$$

Thus condition (A′) is satisfied.

For condition (D_1) we have by (6), for fixed i,

$$(27)\qquad \sum_{j=q}^{\infty} j^{r-1}\left|\Delta_{rr}\phi_{ij}(z, w)\right|$$

$$= \sum_{j=q}^{\infty} j^{r-1}\left|\sum_{s_1=0}^{r}\sum_{s_2=0}^{r}(-1)^{s_1}(-1)^{s_2}\binom{r}{s_1}\binom{r}{s_2}\phi_{i+s_1, j+s_2}(z, w)\right|$$

$$\leqq \left|\sum_{s_1=0}^{r}(-1)^{s_1}\binom{r}{s_1}z^{f(i+s_1)}\right|\sum_{j=q}^{\infty} j^{r-1}\left|\sum_{s_2=0}^{r}(-1)^{s_2}\binom{r}{s_2}w^{g(j+s_2)}\right|.$$

The first part of this expression is the sum of $(r+1)$ terms, each a continuous function of z; whence

$$\lim_{s\to 1}\left|\sum_{s_1=0}^{r}(-1)^{s_1}\binom{r}{s_1}z^{f(i+s_1)}\right| = 0. \tag{28}$$

Next, if $z\neq 0$, we have by (7),

$$\sum_{j=q}^{\infty} j^{r-1}\left|\sum_{s_2=0}^{r}(-1)^{s_2}\binom{r}{s_2}w^{g(j+s_2)}\right| = \frac{1}{\rho^{f(i)}}\sum_{j=q}^{\infty} j^{r-1}\left|\Delta_{0r}\phi_{ij}(z, w)\right|. \tag{29}$$

By a procedure similar to that followed for (B_1) we now find

$$\frac{1}{\rho^{f(i)}}\sum_{j=q}^{\infty} j^{r-1}\left|\Delta_{0r}\phi_{ij}(z, w)\right| < B_{14}(r!)\tau^{g(q)}\left[g(q+r-1)\right]^{r-1}. \tag{30}$$

This is bounded for $\tau<1$, and fixed q. Therefore

$$\lim_{(z,w)\to(1,1)}\sum_{j=q}^{\infty} j^{r-1}\left|\Delta_{rr}\phi_{ij}(z, w)\right| = 0, \tag{31}$$

and condition (D_1) is satisfied. The argument for (D_2) is exactly similar.

Finally, for condition (E) we have

$$\lim_{(z,w)\to(1,1)}\phi_{ij}(z, w) = \lim_{(z,w)\to(1,1)} z^{f(i)}w^{g(j)} = 1; \tag{32}$$

and this completes the proof of the Theorem.

It will be noted that condition (c), $\log t=\sigma\left[f(t)\right]$, etc., is necessitated by Moore's condition (C), $i^{r-1}j^{r-1}\left|\phi_{ij}(z, w)\right| < M(z, w)$. It insures the convergence of the series $J(z, w)$. If, however, a suitable restriction be placed upon $s_{m_1m_2}$, namely, $s_{m_1,m_2}=O\left[\lambda^{f(m_1)+g(m_2)}\right]$ for every $\lambda>1$, condition (c) may be omitted. We may then have $f(t)=\log t$, $g(t)=\log t$, or even more slowly increasing functions. The proof, however, is somewhat long. It will be observed that the Theorem can be extended in an obvious way to multiple series of order n.

HOBART COLLEGE

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

THIRTY-NINTH SUMMER MEETING, CHICAGO, June 19-24, 1933.

This meeting will be held as otherwise previously announced.

NEW YORK CITY, October 28, 1933.

Abstracts must be in the hands of Associate Secretary J. R. Kline, 501 West 116th St., New York City, not later than October 7. Professor R. L. Jeffery will deliver an address on *Theories of integration*. Abstracts received by the Secretaries by September 9 will appear in the September issue of the BULLETIN.

CINCINNATI, OHIO, December 1-2, 1933.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wis., not later than November 11. Abstracts received by the Secretaries by November 4 will appear in the November issue of the BULLETIN.

PASADENA, CALIFORNIA, December 2, 1933.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, Calif., not later than November 11. Abstracts received by the Secretaries by November 4 will appear in the November issue of the BULLETIN.

CAMBRIDGE, MASSACHUSETTS, ANNUAL MEETING, December 27-29, 1933.

This meeting is to be held in conjunction with the A.A.A.S. and the M.A.A. There will be a symposium on *General Analysis*. The Bôcher Prize will be awarded. Abstracts must be in the hands of the Secretary of the Society, 501 West 116th St., New York City, not later than November 30, 1933. Abstracts received by the Secretaries by November 4 will be published in the November issue of the BULLETIN.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or 501 West 116th St., New York.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, September, 1932, p. 66), are payable to the Treasurer of the Society, Professor G. W. Mullins, 501 West 116th St., New York City.

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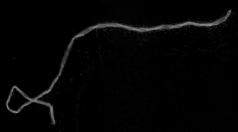
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